Integral Calculus

What is integral calculus?

In differential calculus we differentiate a function to obtain another function called *derivative*. Integral calculus is concerned with the *opposite process*. Reversing the process of differentiation and finding the original function from the derivative is called *integration* or *anti-differentiation*.

The Indefinite Integral

If F(x) is a function of x and its derivative,

$$\frac{d[F(x)]}{dx} = F'(x) = f(x)$$

then F(x) is called an indefinite integral or simply an integral of $f(x)$. Symbolically this is written as,

$$\int f(x)dx = F(x) + c$$

- The symbol $\int$ is the integral sign, *f(x)* is integrand (අනුකලයය) and *c* is the constant of integration.

- Constant *c* may have different values and accordingly we can have different members of *f(x)dx* family.

For example, if $y = x^4$, the derivative of *y* is $\frac{dy}{dx} = 4x^3$

- When $\frac{dy}{dx} = 4x^3$ is given, to find *y* we have to follow the opposite process of differentiation. Thus, $\int 4x^3 dx = x^4$

However, $\frac{dy}{dx} = 4x^3$ may be the derivative of different differentiable functions of y.

For example it is equivalent to the derivatives of

$$y = x^4+1,\ \ y = x^4 - 5,\ y = x^4 + a \quad \text{etc.}$$

- Thus, if we add a constant (c) to the integral it will match with the differential function.

$$\therefore \qquad \int 4x^3 dx = x^4 + c$$

Rules for Indefinite Integration

Rule 1: the Power Rule

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \qquad ; n \neq -1$$

eg. $$\int x^5 dx = \frac{1}{6} x^6 + c$$

Rule 2. the Integral of a Multiple

$$\int kf(x)dx = k\int f(x)dx$$; k is a constant

eg. $$\int 2x^3dx = 2\int x^3dx = \frac{1}{2}x^4 + c$$

Rule 3.the Integral of a Sum or Difference

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

$$\int (2x^2 - 5x + 3)dx = 2\int x^2dx - 5\int xdx + 3\int dx$$

$$= \frac{2}{3}x^3 - \frac{5}{2}x^2 + 3x + c$$

Rule 4. Generalized Power Function Rule

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a} + c$$

eg. i. $\int (4x+2)^{1/2} dx = \dfrac{(4x+2)^{3/2}}{\frac{3}{2}\times 4} = \dfrac{1}{6}(4x+2)^{3/2} + c$

eg. ii. $\int (x^3+1)^2 dx = \dfrac{(x^3+1)^3}{3\times 3x^2} + c = \dfrac{1}{9x^2}(x^3+1)^3 + c$

Rule 5. the Logarithmic Rule

$$\int \frac{1}{x}\,dx = ln|x| \qquad\qquad x \neq 0$$

i. $$\int \frac{3}{x}\,dx = 3\int \frac{1}{x}\,dx = 3\ln|x| + c$$

ii. $$\int \frac{3}{2x-1}\,dx = 3\int \frac{1}{2x-1}\,dx = \frac{3}{2}\ln\,|2x-1| + \mathrm{c}$$

Rule 5. If $f(x)$ is a function of x and its differential coefficient with respect to x is $f'(x)$. The derivative of $\ln[\pm f(x)]$ is

$$\frac{d \ln[\pm f(x)]}{dx} = \frac{f'(x)}{f(x)}$$

$$\therefore \quad \int \frac{f'(x)}{f(x)} dx = \ln\left| f(x) \right|$$

eg. i. $\int \frac{2x}{x^2+1}dx = \ln\left|x^2+1\right|$

eg. ii. $\int \frac{x^2}{x^3+1}dx$

The derivative of the denominator is $3x^2$, and to apply the above rule the numerator should be multiplied by 3 as such denominator, too.

$$\int \frac{3x^2}{(x^3+1)3}dx = \frac{1}{3}\int \frac{3x^2}{x^3+1}dx$$

$$= \frac{1}{3}\ln\left|\mathrm{x}^3+1\right| + \mathrm{c}$$

Rule 6: the Exponential Rule

i. $$\int e^x dx = e^x + c$$

eg. i. $$\int (e^{4x} - 3x^2 + 2e)dx = \int e^{4x}dx - 3\int x^2 dx + 2e\int dx$$

$$= \frac{1}{4}e^{4x} - x^3 + 2ex + c$$

ii. $$\int e^{f(x)}dx = \frac{e^{f(x)}}{f'(x)} + c$$

eg. i. $$\int e^{4x+3}dx = \frac{1}{4}e^{4x+3} + c$$

iii. $$When\ \ y = e^{f(x)}, \quad \frac{d[e^{f(x)}]}{dx} = e^{f(x)} f'(x)$$

$$\therefore \qquad \int e^{f(x)} f'(x)dx = e^{f(x)} + c$$

eg. i. Find indefinite integral of $\int xe^{x^2}dx$

$f(x) = x^2$ and $f'(x) = 2x$

To apply the above rule we have to adjust the original function as,

$$\frac{1}{2}\int 2xe^{x^2}dx$$

There is not a fundamental difference between the original function and this function.

We can apply the above rule for this function

$$\int xe^{x^2}dx = \frac{1}{2}\int 2xe^{x^2}dx = e^{x^2} + c$$

Rule 8: the Substitution rule

$$\int 2x(x^2+1)dx$$

Let $u = x^2+1$; then $du/dx = 2x$ or $dx = du/2x$.

Now $du/2x$ can be substituted for dx of the above function.

$$\therefore \quad \int 2x(x^2+1)dx = \int 2xu\frac{du}{2x} = \int udu$$

$$= \frac{1}{2}u^2 + c$$

$$= \frac{1}{2}(x^2+1)^2 + c$$

$$= \frac{1}{2}(x^4+2x^2+1) + c$$

eg. $$\int 2x(x^2+8)^3\,dx$$

If we defined u = x^2+ 8 and du/dx = 2x. From this

$$\frac{du}{2x} = dx$$

$$\int 2x(x^2+8)^3\,dx = \int 2xu^3\,dx = \int 2xu^3\,\frac{du}{2x}$$

$$= \int u^3\,du = \frac{1}{4}u^4$$

$$= \frac{1}{4}(x^2+8)^4 + c$$

The Definite Integral

The indefinite integral of a continues function *f(x)* is:

$$\int f(x)dx = F(x) + c$$

If we choose two values of *x* in the domain, say *a* and *b* (b > a), substitute them successively into the right side of the above equation and form the difference we get a numerical value that is independent of the constant c.

$$[F(b)+c]-[F(a)+c] = F(b)-F(a)$$

This value is called the definite integral of *f*(x) from a to b.
a and b are lower and upper limits of integration, respectively.

Now, we will modify the integration sign to indicate the definite integral of *f*(x) from a to b as:

$$\int_a^b f(x)dx = [F(x)+c]_a^b = [F(b)+c]-[F(a)+c]$$
$$= F(b)-F(a)$$

Evaluate the following definite integrals

1. $\int_2^4 3x^2 dx = 3\left[\frac{x^3}{3}\right]_2^4 = 4^3 - 2^3 = 56$

2. $\int_1^2 (6x^2 + 8x + 1)dx = 27$

3.

$$\int_0^2 (x+7)^3\,dx = \left[\frac{(x+7)^4}{4}\right]_0^2$$

$$= \frac{1}{4}\left[(2+7)^4 - (0+7)^4\right]$$

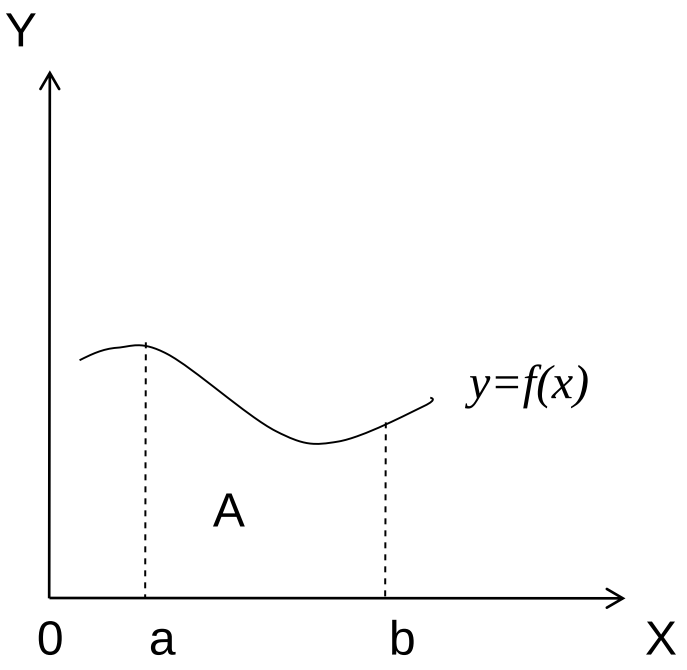
$$= \frac{1}{4}\left(9^4 - 7^4\right)$$

$$= \frac{1}{4}(4160) = 1040$$

4. $$\int_a^b ke^x\,dx = ke^x\Big]_a^b = k(e^b - e^a)$$

The Definite Integral as an Area

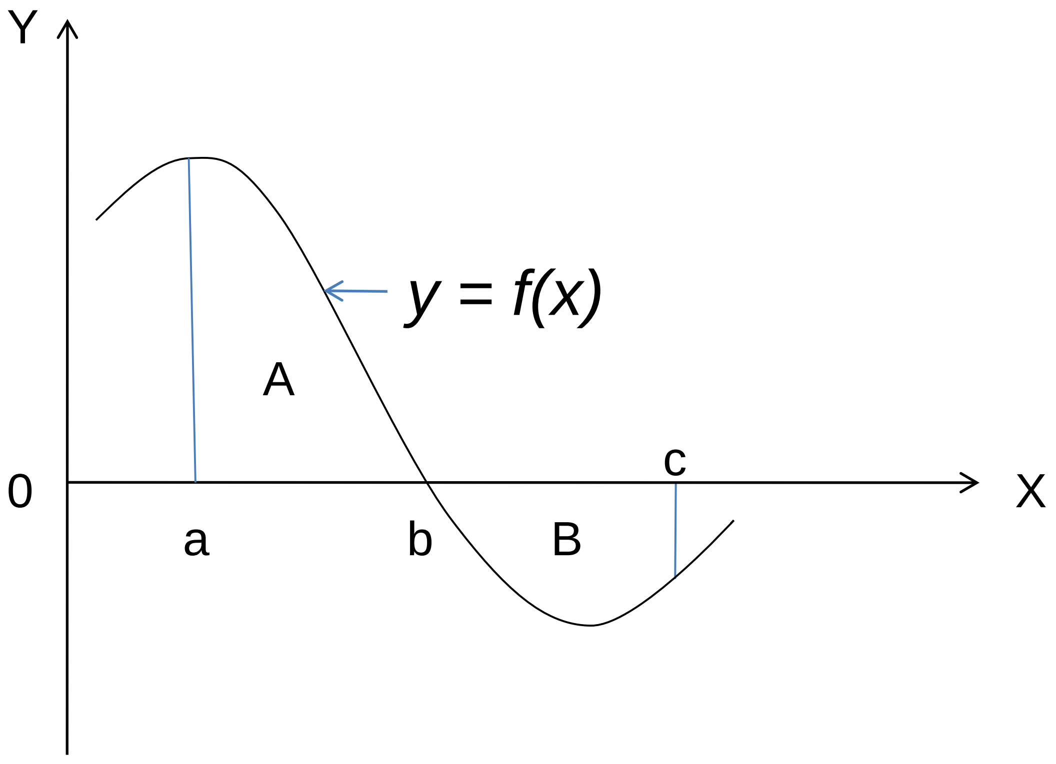
The area of the region bounded by the curve y = *f*(x), and by the x axis, on the left by $x = a$, and on the right by $x = b$ is given by,



$$\text{Area }(A) = \int_a^b f(x)dx$$

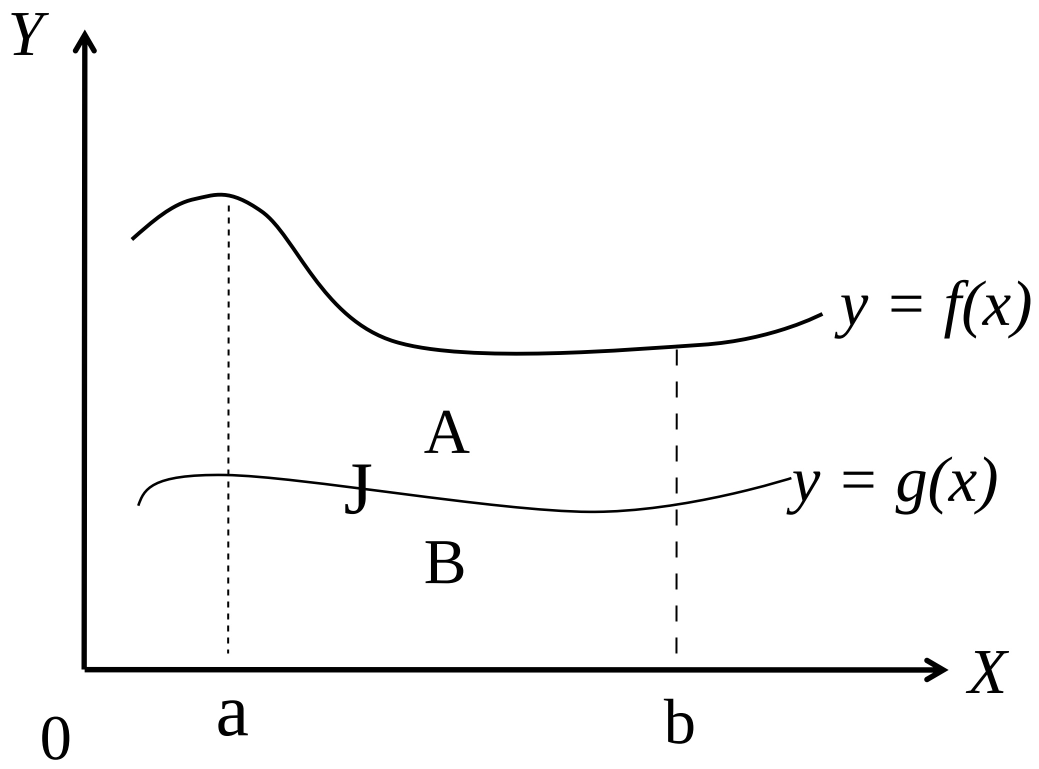
If the curve $y = f(x)$ lies below the x axis, then

$$\text{Area }(A) = -\int_a^b f(x)dx$$



The area (A) + B) = $\int_{a}^{b} f(x)dx + \int_{b}^{c} f(x)dx$

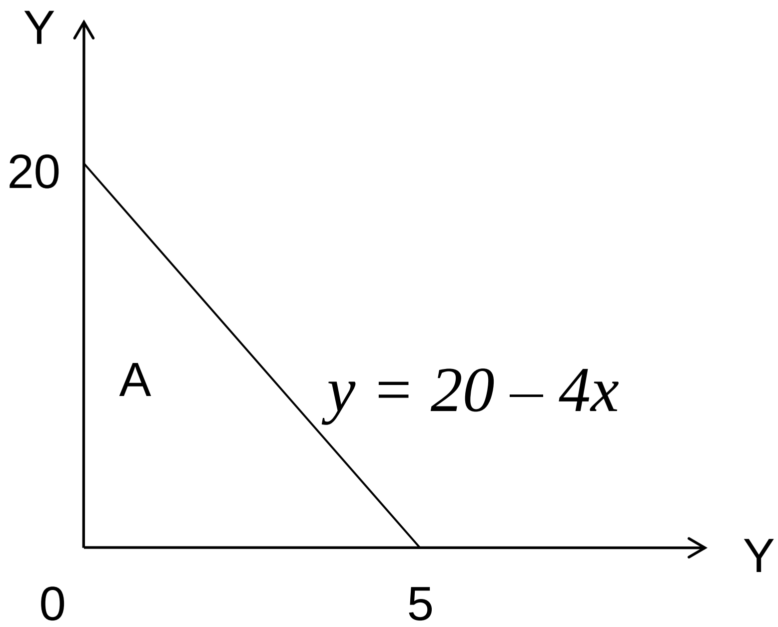
IF *f(x)* and *g(x)* are the two functions of *x* and *f(x)* > *g(x)*



Area (A) = Area (J) – Area (B)

$$= \int_a^b f(x)dx - \int_a^b g(x)dx$$

eg.1. Determine the area under the curve given by the function   y = 20 – 4x over the interval 0 to 5.

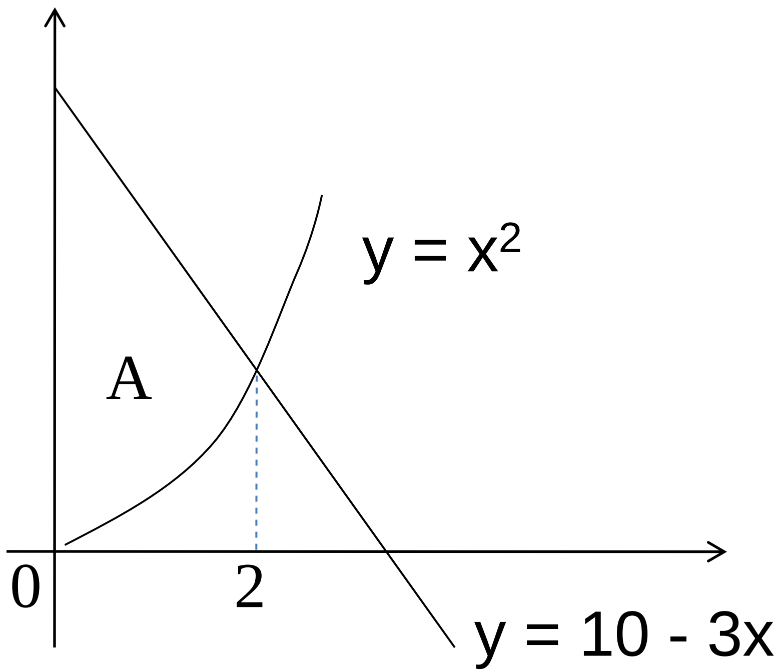


$$\text{Area (A)} = \int_0^5 (20-4x)dx = \left[20x - 2x^2\right]_0^5$$

$$= (100 - 50) - (0)$$

$$= 50$$

Eg. 2 Find the area bounded by the functions $y = x^2$, and $y = 10 - 3x$ and Y axis.

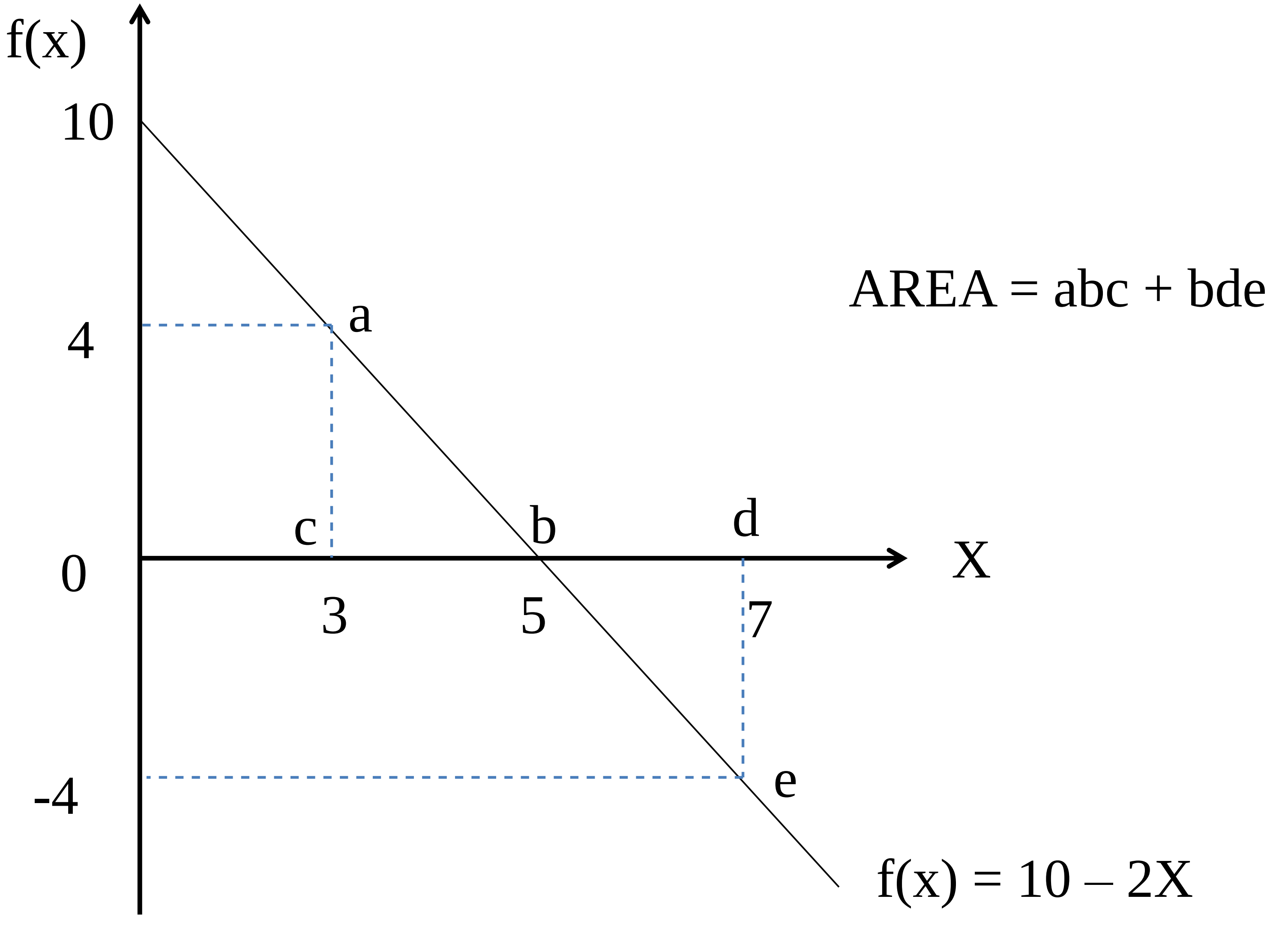


$$A = \int_0^2 (10 - 3x)dx - \int_0^2 x^2 dx$$

$$= \left[10x - \frac{3}{2}x^2\right]_0^2 - \left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{34}{3}$$

3. Determine the area between the curve of $f(x) = 10 - 2X$ and the X axis for values of X = 3 to X = 7.



$$\text{AREA a}bc = \int_3^5 (10 - 2X)dX = [10X - X^2]_3^5$$

$$= 4$$

$$\text{AREA bde} = \int_5^7 (10 - 2X)dX = [10X - X^2]_5^7$$

$$= -4$$

$\therefore$ AREA = abc + bde = 8

Economic Applications

1. If the marginal cost (MC) function of a firm is $C' = 2e^{0.2Q}$ and the fixed cost C_F = 90. Find the total cost function (TC).

2. If the marginal cost (MC) function of a firm is $MC(q) = (6q^2 + 4)\sqrt{2q^3 + 4q + 36}$ and fixed cost of the function is 1088. Find the total cost function (TC).

3. If the marginal saving function of a country is $S'(Y) = 0.3 - 0.1Y^{-1/2}$. If the aggregate saving S is zero when income (Y) is 81. Find the saving function S(Y).

4. Consumer's demand function for a given commodity has been estimated to be $P = 30 - 2Q$

where, P is the price of a unit of the commodity and Q is the per capita consumption of the commodity per person per month. Determine (a) the total expenditure and (b) the consumer surplus when the price of a unit is 5.

5. If the supply function of a commodity is $P = 1000 + 50Q$ where, P is the price per unit and Q is the number of units sold each day. Find the producer surplus when the price of a unit of the commodity is 2000.

6. If the willingness of a nurse to provide her service is defined by the supply function $W = 2.5 + 0.5H$

where, W is the wage rate per unit
H is hours of work provided each week.

Determine the producer surplus paid to the nurse if the prevailing wage rate is 9 per hour.

9. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t) = 50 + t^2$ hundred dollars per year, while a second investment will be generating profit at the rate of $P_2'(t) = 200 + 5t$ hundred dollars per year. $P1(t)$ and $P2(t)$, satisfy $P_2(t) \geq P_1(t)$ for the first N years ($0 \leq t \leq N$).

(a) For how many years does the rate of profitability of the second investment exceed that of the first?

(b) Compute the net excess profit for the time period determined in part (a). Interpret the net excess profit as an area.

10. Suppose that when it is t years old, a particular industrial machine generates revenue at the rate $R(t) = 5{,}000 - 20t^2$ dollars per year and that operating and servicing costs related to the machine accumulate at the rate $C(t)$ $2{,}000 + 10t^2$ dollars per year

(a) How many years pass before the profitability of the machine begins to decline?
(b) Compute the net earnings generated by the machine over the time period determined in part (a)

12. එක්තරා භාණ්ඩයක් සඳහා වූ පරිභෝජක ඉල්ලුම් ශ්‍රිතය

$D(q) = 4(25-q^2)$ වේ.

i. භාණ්ඩයෙන් ඒකක 3ක් මිලට ගැනීම සඳහා පරිභෝජකයා ගෙවීමට සූදානම් මුදල් ප්‍රමාණය කොපමණ ද?

13. මෝටර් රථ ටයර නිෂ්පාදන සමාගමක් ඇස්තමේන්තු කර ඇති ආකාරය අනුව ගැනුම්කරුවන් විසින් ඒකකයක මිල රුපියල් $D(q)= -0.1q^2 + 90$ වන විට ටයර q (දහස්) ප්‍රමාණයක් ඉල්ලුම් කරන අතර ආයතනයඑම ප්‍රමාණය සපයන්නේ $S(q) = 0.2q2 + q + 50$. මිලට ය.

i. සමතුලිත මිල හා ප්‍රමාණය ගණනය කරන්න.

ii. සමතුලිතයේදී පරිභෝජක අතිරික්තය හා නිෂ්පාදක අතිරික්තය ගණනය කරන්න.