Mathematics for Economics
ECON 53035
MA/MSSc in Economics-2017/2018

Dr. W. M. Semasinghe
Senior Lecturer
Department of Economics

MATHEMATICS AND STATISTICS

LERNING OUTCOMES:

By the end of this course unit students will be able to demonstrate skills in mathematical and statistical methods that are highly useful in analyzing problems related to economic theory and practice and understand the uses of basic descriptive and inferential statistics in economic analysis.

COURSE CONTENTS: This course unit consists of two parts:

Part I: Mathematics - Functions and their applications in economics, Calculus and its applications in economics: Differentiation, Partial differentiation, Integration; Matrices; Maxima and Minima; Constrained Optimization with economic applications, Linear programming.

Part II: Statistics - Probability and probability Theory; continuous and discrete variables, continuous and discrete variables distributions, Joint distributions; Moment generating functions; Hypothesis testing and confidence intervals.

Functions

- A function is a rule that gives exactly one output number to each input number.

Why it is important to us?

If the value of variable ***y*** depends on the values of another variable say ***x***, *y* is said to be a function of *x* and, denoted by,

$$y = f(x)$$

Where,

$y \rightarrow$ dependent variable or value of the function

$x \rightarrow$ independent variable or argument of the function

$f \rightarrow$ particular rule that applies eg:- $q = f(p)$ $s = f(y_d)$

$p = f(q)$ $c = f\,(y_d)$

$TC = f(q)$ $TR = f(q)$

To denote a function the symbols F, G, g, h, and the Greek letters such as θ, φ, ψ, Φ, Ψ etc. are also used.

If y and z depend on x, then $y = f(x)$ and $z = f(x)$
We can also write
$y = y(x)$
$z = z(x)$

- ❖ The set of all input numbers to which the rule applies is called the ***domain*** of the function.

- ❖The set of all output numbers is called the ***range***.

$y = 18x - 3x^2$

If the input (domain) x is any real number, then the output (range) y is also a real number, R.

$$y = \frac{2x + 3}{x - 1}$$

The output ***y*** is a real number if the input ***x*** is any real number other than 1.

the domain is $R - \{1\}$

Since, most variables in economic models are by their nature restricted to nonnegative real numbers, their domains are also restricted.

eg. The total cost of a firm per day is a function of its daily output Q: C=150+7Q. The firm has a capacity limit of 100 units of output per day.
What are the domain and rang of the cost function?

Domain $= \{ Q \mid 0 \leq Q \leq 100 \}$

Range $= \{ C \mid 150 \leq C \leq 850 \}$

- If ‘a’ is any particular value of the function of x, the value of function $f(x)$ for x = a is denoted by f(a).

$$y = f(x) = \frac{x}{7x + 1}$$

$$f(a) = a/(7a + 1)$$

$$f(4) = 4/\ 29$$

Types of Functions

- Constant function: A function whose range consists of only one element.

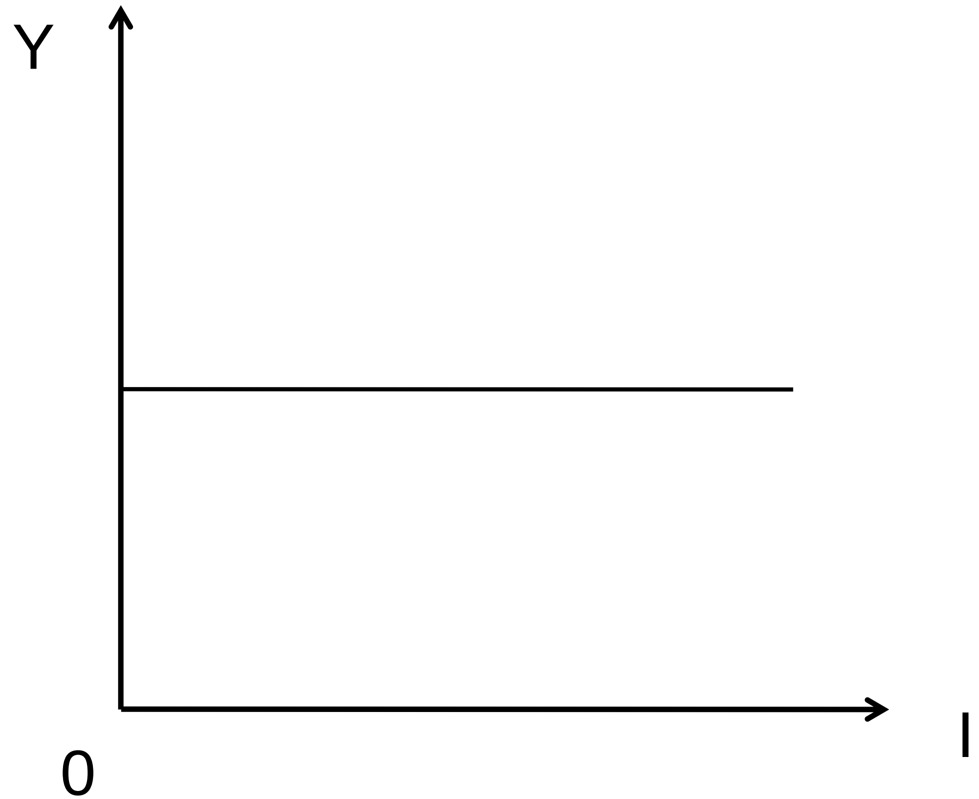
eg. $y = 7$

$f(x) = 10$

The value of the function does not change whatever the value of x.

In a coordinate plane, such a function will appears as a horizontal straight line.

eg. In a national income model, the investment function which determined exogenously



❖ Polynomial Functions

The general form of a Polynomial Functions of Degree n

$$\boldsymbol{f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_{n-1}x^{n-1} + a_nx^n}$$

n is a positive integer and $a_0 \ldots\ldots a_n$ are constants. $a_n \neq 0$

e.g. $f(x) = 8x^6 + 3x^4 - x^3 + 5x^2 + 2x + 3$

(polynomial of degree 6)

$F(x) = x^8 + 2x^5 + 3x^4 + 7x^2 + 6x - 5$

(polynomial of degree 8)

Depending on the value of the integer n there are several subclasses of polynomial functions.

when

$n = 0$	$y = a_0$	Constant function
$n = 1$	$y = a_0 + a_1x$	Linear function (polynomial of degree 1)
$n = 2$	$y = a_0 + a_1x + a_2x^2$	Quadratic function
$n = 3$	$y = a_0 + a_1x + a_2x^2 + a_3x^3$	Cubic function

❖ Rational Functions

$$f(x) = \frac{g(x)}{h(x)}$$

g(x) and h(x) are polynomials of x and $h(x) \neq 0$

f(x) is expressed as a ratio of two polynomial functions of variable x.

$$f(x) = \frac{(3x^2 + 5)}{(2x - 1)}$$

❖ Composite Function or Function of a Function

If $y = g(u)$ and $u = f(x)$, $y = g[f(x)]$

eg. 1. If $y = u^2 + 3$ and $u = 2x + 1$ then,

$$y = (2x+1)^2 + 3$$

eg. 2. If $y = x^3 - 3x + 5$ and $x = \frac{1}{2}\sqrt{t} + 3$ then,

$$y = ?$$

$$y = (\tfrac{1}{2}\sqrt{t}+3)^3 - 3(\tfrac{1}{2}\sqrt{t}+3) + 5$$

❖ Non-algebraic Function

The function such as

$$y = e^t$$

$$y = b^x$$

which the independent variable appears in the exponent is called non-algebraic function.

Logarithmic functions are also non- algebraic functions.

$$y = \log_b x$$

$$y = \ln x$$

Functions of two or more independent variable

$$z = f(x, y)$$

$$z = ax + by$$

$$z = a_0 + a_1x + a_2x^2 + b_1y + b_2y^2$$

$$y = a_1x_1 + a_2x_2 + a_3x_3 + \ldots a_nx_n$$

$$V = AL^{\alpha}K^{\beta}$$

Differentiation

The process of finding the derivative (deferential coefficient) of a function is called *differentiation*. It stands for finding the rate of change of one variable, say y, with respect to the rate of change of another variable, say x.

For a function y = *f*(x) the differential coefficient of y with respect to x is defined as

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$y = f(x) \qquad (1)$$

Assume, when variable x is increased by Δx, variable y will be increased by Δy,

$$y + \Delta y = f(x + \Delta x) \qquad (2)$$

(2) - (1) $$\Delta y = f(x + \Delta x) - f(x) \qquad (3)$$

(3) $\div \Delta x$ $$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The limit of $\Delta y/\Delta x$ when $\Delta x \xrightarrow{\lim} 0$, is defined as the *differential coefficient* of $f(x)$

- dy/dx is used to denote the limiting value $\Delta y/\Delta x$

eg. $y = 3x$ ………………….. (1)

$y + \Delta y = 3(x + \Delta x)$ …………(2)

(2) – (1) $\Delta y = 3(x + \Delta x) - 3x$

$\Delta y = 3\Delta x$ …………(3)

(3) ÷ Δx $\Delta y/\Delta x = 3$

$$\lim_{\Delta x \to 0} \Delta y/\Delta x = 3$$

Rules of Differentiation

When u, v and y are the functions of x and *k* is a constant,

(1) Constant function rule

$$y = f(x) = k$$

$$\frac{dy}{dx} = \frac{d(k)}{dx} = f'(x) = 0$$

(2) **Linear function rule**

$$y = f(x) = kx + 2$$

$$dy/dx = k$$

(3) Power function rule

$$y = f(x) = x^n$$

$$\frac{dy}{dx} = \frac{d(x^n)}{dx} = nx^{n-1}$$

(4). Generalized power function rule

$$y = kx^n$$

$$\frac{dy}{dx} = \frac{d(kx^n)}{dx} = knx^{n-1}$$

(5). Sum-difference rule

$y = [f(x) \pm g(x)]$

$$\frac{dy}{dx} = \frac{d[f(x)]}{dx} \pm \frac{d[g(x)]}{dx}$$

(6). Function of a function rule

When $y = [f(x)]^n$

This type of a function is called function of a function.

$$\frac{dy}{dx} = n[f(x)]^{n-1} f'(x)$$

(7). Product rule

$$y = [f(x).g(x)]$$

$$\frac{dy}{dx} = f(x) \frac{d[g(x)]}{dx} + g(x) \frac{d[f(x)]}{dx}$$

$f(x) = u$ and $g(x) = v$

$$y = uv$$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

If u, v and w are functions of x,

$$\frac{d\,(u,v,w)}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

eg. $y = x^2\,(x^2 + 1)\,(x + 2)$

(8). Quotient rule

$$y = \frac{f(x)}{g(x)}$$

$$\frac{dy}{dx} = \frac{g(x)\dfrac{df(x)}{dx} - f(x)\dfrac{dg(x)}{dx}}{[g(x)]^2}$$

$f(x) = u \qquad g(x) = v$

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

When $y = 1/f(x)$

$$\frac{dy}{dx} = \frac{-\ f'(x)}{[f(x)]^2}$$

eg. $y = 1/\ (x^2+1)$

$$\frac{dy}{dx} = \frac{-2x}{\left(x^2+1\right)^2}$$

(9). Chain rule

When $y = f(u)$ and $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

1. Find dy/dx given $y = \dfrac{u^2 - 1}{u^2 + 1}$ and $u = \sqrt[3]{x^2 + 1}$

(10). Log-function rule

When a variable is expressed as a function of logarithm of another variable, the function is referred to as a *logarithmic function* or *log function.*

$y = \log_b x$ $\qquad$ $y = \log_e x \ (= \ln x)$

Which differ from each other only in regards to the base of logarithm.

In calculus *e* takes as the base of logarithm.

- Logarithms on base *e* is called *natural logarithm* and denoted as $\log_e$ or ln.

eg. $\quad y = \log_e 2x$ or $\quad y = \ln 2x$

1. $y = \ln x$

$$\frac{dy}{dx} = \frac{d(\ln x)}{dx} = \frac{1}{x}$$

2. $y = k\ln x$

$$\frac{dy}{dx} = k \frac{d(\ln x)}{dx} = k\frac{1}{x} = \frac{k}{x}$$

3. If $y = \log_e u$ and $u = f(x)$

$$\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx}$$

(11). Exponential Function Rule

- Power (index) of the independent variable in a polynomial function is called *exponent.*

eg. $y = 2x^3 + 4x^2 + 5$

- The functions which appears independent variable as the exponent is called *Exponential function*.

eg. $y = f(x) = b^x \quad (b>1)$

- where, y and x are dependent and independent variables respectively and b is the *base* of the exponent.

In calculus, an irrational number e (e = 2.71828….) takes as the base of the exponent.

eg. $y = e^x \quad y = e^{3x} \quad y = Ae^{rx}$

$$y = e^x$$

$$\frac{dy}{dx} = e^x$$

eg. If $y = e^u$ and $u = f(x)$

$$\frac{dy}{dx} = e^u \frac{du}{dx}$$

eg. $y = e^{(3x + 2)}$

$$\frac{dy}{dx} = 3e^{(3x + 2)}$$

Derivatives of Higher Order

When derivative of a function is differentiable, it may have a derivative as well.

If $y = f(x)$, then dy/dx or $f'(x)$ is called 1st derivative of the function.

The derivative of dy/dx or $f'(x)$ is denoted by d^2y/dx^2 or $f''(x)$ and is called the second derivative.

If $y = f(x)$

$$1^{st}\text{ derivative: } \quad f'(x) = \frac{d\,(y)}{dx} = \frac{dy}{dx}$$

$$2^{nd}\text{ derivative: } \quad f''(x) = \frac{d\,(dy/dx)}{dx} = \frac{d^2y}{dx^2}$$

$$3^{rd} \text{ derivative: } f'''(x) = \frac{d\,(d^2y/dx^2)}{dx} = \frac{d^3y}{dx^3}$$

$$n^{th} \text{ derivative: } f^{\,n}(x) = \frac{d\,(d^{n-1}y/dx^{n-1})}{dx} = \frac{d^ny}{dx^n}$$

Find

i. $y = x^4 + 3x^2 - 2x + 7$ 3^{rd}

ii. $y = x\sqrt{1 - 2x}$ 2^{nd}

ii. $y = \dfrac{3x - 1}{x + 2}$ 3^{rd}

Partial differentiation

So far we dealt with functions of a single independent variable x, which were generally expressed as $y = f(x)$. However, there are many situations in which we must consider more than one independent variables.

A function of two independent variables x and y can be expressed as

$$z = f(x, y).$$

Z is the dependent variable.

In general, if *y* is a function of *n* independent variables $x_1, x_2, \ldots x_n$, it can be written as

$y = f(x_1, x_2, \ldots., x_n)$

Given a multivariable function $y = f(x_1, x_2, \ldots., x_n)$, the *partial derivative* measures the rate of change of the dependent variable (y) with respect to change of one of the independent variables, while the other independent variables are held constant.

If $z = f(x, y)$, the partial derivative of $\boldsymbol{z}$ with respect to x is obtained by treating $\boldsymbol{y}$ as a constant and applying ordinary rules of differentiation.

The partial derivative of $\boldsymbol{z}$ with respect to $\boldsymbol{x}$ is symbolized by $\partial \boldsymbol{z}/\partial \boldsymbol{x}$ or f_x.

Similarly, the partial derivative of $\boldsymbol{z}$ with respect to $\boldsymbol{y}$ is symbolized by $\partial \boldsymbol{z}/\partial \boldsymbol{y}$ or $\boldsymbol{f_y}$.

eg. Find the partial derivatives (f_x and f_y) of the following functions:

i. $f(x_1,x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$ Find f_x and f_y

ii. $f(x, y) = (x^2 – 7y) (x – 2)$ Find f_x and f_y

iii. $f(x,y) = (2x – 3y)/(x + y)$ Find f_x and f_y

iv. $z = \ln (x^2 + y^2)$

v. $z = (x + y) e^{(x + y)}$

vi. $z = 3x^2y + 4xy^2 + 6xy$

Eg. 2. The indifference curve of a consumer is given as

$$U = U(x_1, x_2) = (x_1 + 2)^2 (x_2 + 3)^2$$

Where, U is the total utility, x_1 and x_2 are the quantity consumed from the two goods X and Y.

i. Derive the marginal utility functions of the two goods.

ii. If consumer buys 3 items from each good, find the marginal utility of 1st good.

Eg. 3. Derive the marginal productivity functions of labor (MP_L) and capital (MP_K) from the production function

$$V = AL^{\alpha}K^{\beta} .$$

Where, V is the output, L and K are labor and capital inputs respectively.

Partial Derivatives of Higher Order

- Higher partial derivatives are obtained in the same way as higher derivatives.

- For the function z = f(x, y), there are 4 second order partial derivatives:

i $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial\,(\partial z/\partial x)}{\partial x} = (f_x)_x = f_{xx}$$

ii $$\frac{\partial^2 z}{\partial y^2} = \frac{\partial(\partial z/\partial y)}{\partial y} = (f_y)_y = f_{yy}$$

$$\text{iii} \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial \ (\partial z/\partial x)}{\partial y} = (f_x)_y = f_{xy}$$

$$\text{iv} \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial(\partial z/\partial y)}{\partial x} = (f_y)_x = f_{yx}$$

Cross partial derivatives

Find all four partial derivatives of higher order

$$z = 4x^6 - 3x^2y^2 + 5y^4$$

$$z = x^2e^{-y}$$

$$z = 7x \ln(1 + y)$$

$$z = (2x + 5y)(7x - 3y)$$

$$z = e^{4x - 7y}$$

Optimization

- Optimization is the choice of the best among the available alternatives.

Optimization

Maximization

- Profit
- Utility
- Growth rate
- Sales
- Output etc.

Minimization

- Cost
- Risk etc.

$y = f(x)$

This is called objective function, if the aim is to find the value of x which optimizes the value of dependent variable y. It may be a maximization or a minimization.

x – choice variable

- decision variable

- policy variable

This indicates the value that it should take to maximize/minimize variable y.

Suppose a business firm seeks to maximize profit (π). Profit is maximized at the output (Q) which maximize the difference between total revenue (R) and total cost (C). Given state of technology and market demand, both R and C are functions of Q. There for π is also a function of Q.

$$\pi = R(Q) - C(Q)$$

π indicates the goal of the firm i.e. what should optimize. Thus, it is the objective function of the optimization problem. Q is the choice variable. The aim is to find the Q that maximize the π.

Determination of maxima and minima of a function

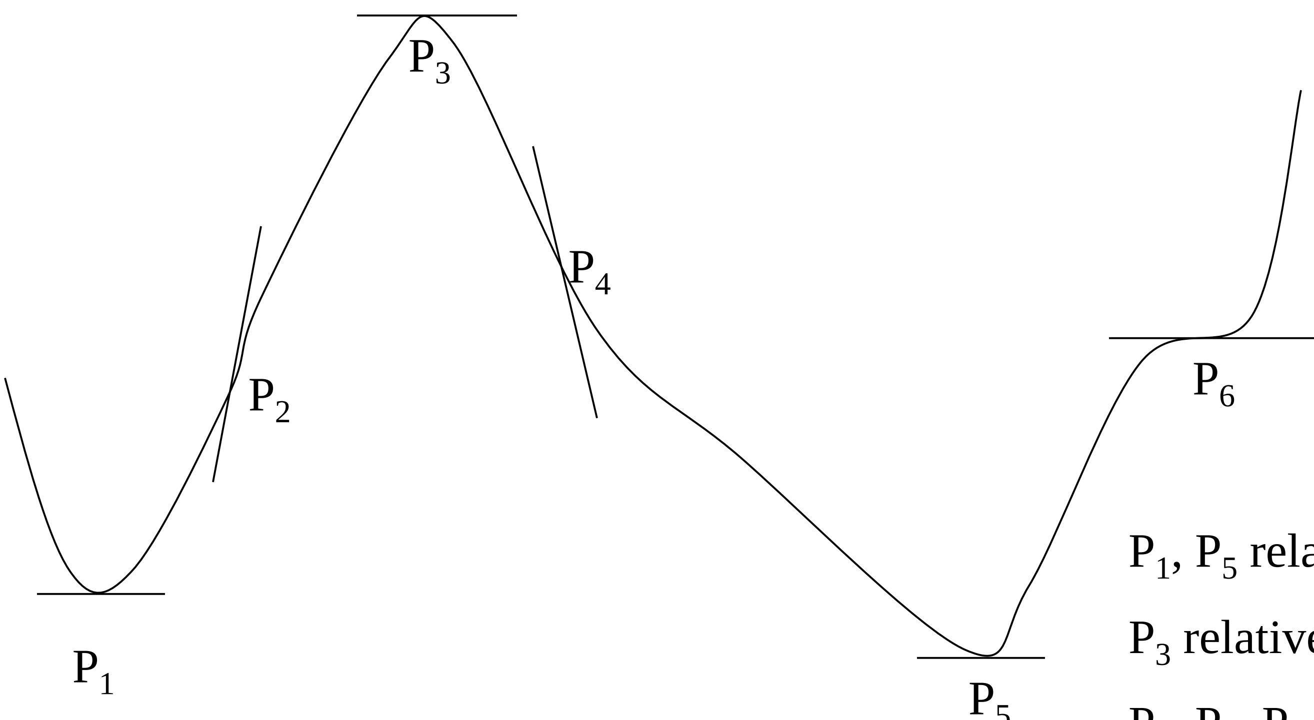
❖ A point on a graph that is higher than any other point in its vicinity is called a ***relative maximum***. In other words, $y = f(x)$ is said to have a relative maximum value at $x = x_0$ if $f(x_0)$ is greater than immediately preceding or succeeding values of the function.

❖ Similarly, a point on a graph that is lower than any other point in its vicinity is called a ***relative minimum***. In other words, $y=f(x)$ is said to have a relative minimum value at $x = x_0$ if $f(x_0)$ is smaller than immediately preceding or succeeding values of the function.

- ❖ An extreme point (relative or local extremum) of a function is a point where the function is at a relative maximum or minimum. If the derivative of the function exists at an extreme point, the value of the derivative must be zero, and the tangent line, if exists, is horizontal.

- ❖ A point where the derivative equals zero or undefined is called a ***critical point***. Since the derivative of a function at a relative extreme must be zero, a relative extreme can occur only at a critical point.

❖ A point at which a curve crosses its tangent is called an ***inflection point***. The sign of the first derivative can be positive or negative or can be zero for an inflection point.



P_1, P_5 relative minimum

P_3 relative maximum

P_2, P_4, P_6 inflection points

Determination of maximum and minimum

1. First derivative test

2. Second derivative test

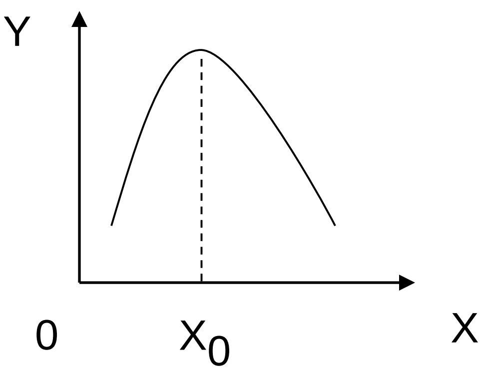
❖ ***First derivative test*** – first derivative of the function is used to determine the extreme points.

Necessary condition or first order condition

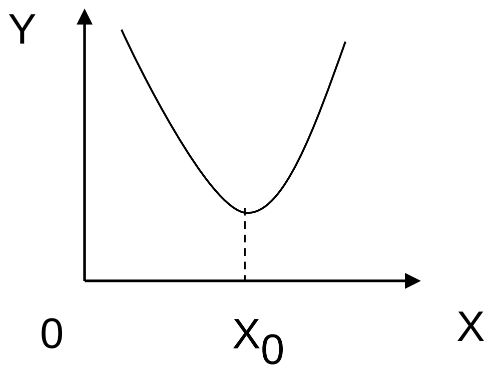
If the first derivative of a function $y = f(x)$ at point $x = x_0$ is zero i.e. $f'(x) = 0$ then it is an extreme point. It may be a relative maximum or minimum or inflection point.

Sufficient condition or second order condition

- If the sign of the derivative $f'(x)$ changes from positive to negative from the left of the point x_0 to its right, x_0 is a relative maximum.



- If the sign of the derivative $f'(x)$ changes from negative to positive from the left of the point x_0 to its right, x_0 is a relative minimum .



- If $f'(x)$ has the same sign on both sides of point x_0, $f(x)$ has neither maximum nor minimum value at $x = x_0$. It is an inflection point.

Second Derivative Test

Second derivative of the function is used to determine the extreme points.

i Solve $f'(x) = 0$ for the critical values.

ii. For a critical value $x = x_0$

$f(x)$ has a maximum value if $f''(x_0) < 0$

$f(x)$ has a minimum value if $f''(x_0) > 0$

However, the test fails if $f''(x_0) = 0$ or becomes infinite?

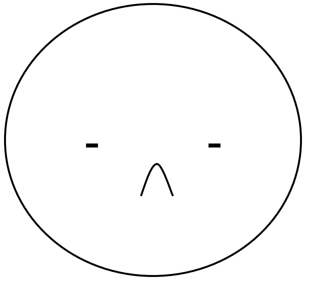
i. If at $x = x_0$ $f'(x_0) = 0$, x_0 is a critical point.

ii. If $f''(x_0) < 0$, x_0 is a maximum point.

Maximum

$dy/dx = f'(x) = 0$

$d^2y/dx^2 = f''(x) < 0$

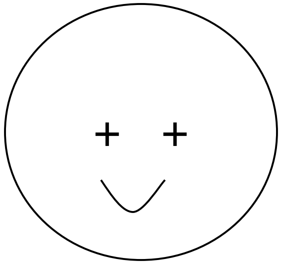


iii. If $f''(x_0) > 0$, x_0 is a minimum point.

Minimum

$dy/dx = f'(x) = 0$

$d^2y/dx^2 = f''(x) > 0$



If $d^2y/dx^2 = f''(x) = 0$?

In a situation like this,

(1) Examine how the sign of $f'(x)$ change when x substitutes values less than to higher than of x_0

i. If the sign changes + to – at $x = 0$, the function has a maximum.

ii. If the sign changes - to + at $x = 0$, the function has a minimum.

iii. If the sign does not change the function has a inflection at $x = 0$

(2). Take successive higher order derivatives.

Substitute $x = x_0$ to the first non-zero higher order derivative.

- If the value is odd, $x = x_0$ is an inflection point.
- If the value is even, $x = x_0$ is an extreme point.

- If the even number is positive, the point is minimum

- If the even number is negative, the point is maximum.

1. The total cost of producing a given commodity is

$TC = \frac{1}{4} x^2 + 35x + 25$ and the price of the commodity is

$P = 50 - \frac{1}{2} x$.

i. Find the level of output which yield the maximum profit

i. Show that average cost is minimum at this output level.

2. Cost function of a perfectly competitive firm is

$$TC = 1/3\ q^3 - 5q^2 + 30q + 10$$

If price p = 6, find the profit maximizing output level.

Extreme Values of a Function of Two Choice Variables

z = f(x, y)

Condition	Maximum	Minimum
First order or Sufficient	$f_x = 0,\ f_y = 0$	$f_x = 0,\ f_y = 0$
Second order or Necessary	$F_{xx}, f_{yy} < 0$ and $f_{xx} f_{yy} > f_{xy}^2$	$f_{xx}, f_{yy} > 0$ and $f_{xx} f_{yy} > f_{xy}^2$
	If $f_{xx} f_{yy} < f_{xy}^2$ *a saddle point*	

Economic applications

Profit maximization

Profit (Π) = Total Revenue (R) – Total Cost (C)

$$\Pi = R - C$$

$$R = f(q) \quad \text{and} \quad C = f(q)$$

$$\Pi(q) = R(q) - C(q)$$

For profit maximization, the difference between R(q) and C(q) should be maximized.

First order condition

$$d\Pi/dq = \Pi' = 0$$

$$\Pi'(q) = R'(q) - C'(q)$$

$$= 0 \text{ iff } R'(q) = C'(q)$$

Thus, at the output level which yield maximum profit,

$$R'(q) = C'(q)$$

$$MR = MC$$

This is the first order condition for profit maximization.

Second order condition

$$d^2\Pi/dq^2 = \Pi''(q) < 0$$

$$\Pi''(q) = R''(q) - C''(q)$$

$$< 0 \quad \text{iff } R''(q) < C''(q)$$

Thus, to satisfy the second order condition $R''(q) < C''(q)$.

$R''(q)$ = the rate of change of MR

$C''(q)$ – the rate of change of MC

$\therefore$ At the output level which MC = MR, and $R''(q) < C''(q)$ profit is maximized.