MA/MSSc in Economics
ECON 50315-
Advanced Economic Theory- Microeconomics

# Theory of Production

Prof. W. M. Semasinghe

# Introduction

Theory of production deals with the question of ‘How to produce?’

It discusses the supply side of the pricing of products.

Supply of a product depends on the cost of production.

Cost of production, in turn depends on,

i. the physical relationship between inputs and output

ii. the prices of inputs.

This physical relationship forms the subject matter of theory of production.

More specifically, theory of production relates to ‘the physical laws governing production of goods’.

## Physical Laws of Theory Production

- The Law of Variable Proportions or the Law of Diminishing Marginal Product – short run behavior

- The Laws of Returns to Scale – Long run behavior

Theory of production concerned with explaining:

how a firm choose the optimum factor combination *either,*

- To minimize the cost of production for given level of output *or*
- To maximize the level of output for a given cost/set of input.

# Production function

The functional relationship between inputs and output is called the 'Production Function'.

Theory of production is the study of production functions.

- *Definition: Production Function is an embodiment of the technology which yields maximum output from the given set of inputs or specifies the way in which inputs co-operate together to produce a given level of output.*

A production function may take the form of a schedule, a graphical line or curve, an algebraic equation or a mathematical model.

The general form of a production function which use n inputs can be algebraically expressed as:

$$q = f(x_2, x_2, .........x_n)$$

Where,

$q$ = flow of output in physical terms

$x_1, x_2, ......x_n$ are flows of inputs in physical terms per unit of time.

Two inputs (L and K) production function:

$$q = f(L, K)$$

Where,

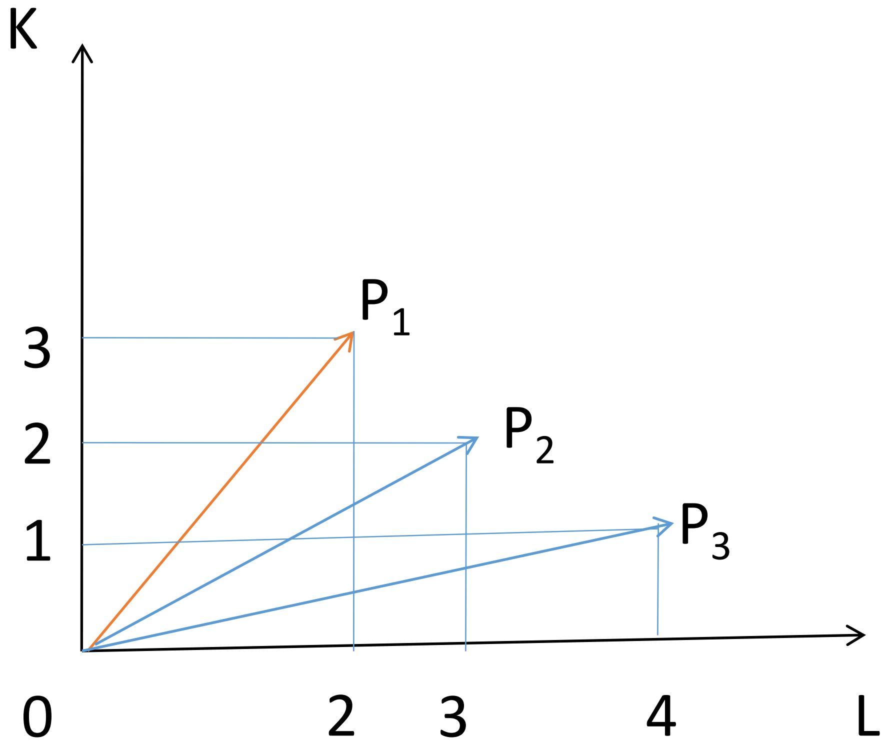
$q$ = flow of output in physical terms

$L$ *and* $K$ = flows of labor and capital inputs in physical terms per unit of time.

- Since it is assumed that inputs can be combined in varying proportions, given output can be produced in many ways. This is the reason for presence of various methods of production.

- This means in relation to the above production function is that the given commodity can be produced by more capital and less labour, and vice versa.

Alternative production processes of $q=f(L, K)$




The figure illustrates three alternative production methods $P_1$, $P_2$ and $P_3$.

- Production function analysis is used widely by the Economists in describing the production process compared with other analysis i.e. *Input-output analysis* and *Linear Programming*, since production function involves and provides measurements on number of concepts which are useful tools in all field of economics.

$$q = f(L, K)$$

## 1. Marginal product of factors of production (MP)

- MP of a factor is the addition in total output by employing one more unit of that factor, other factors remaining unchanged.

- Mathematically MP of a factor is the partial derivative of the production function (q) with respect to that factor.

$$MP_L = \frac{\partial q}{\partial L} \quad and \quad \mathrm{MP_K} = \frac{\partial q}{\partial K}$$

- Graphically it is given by the slope of Total Product Curve.

## 2. Average Product

AP measure the output per unit of a particular input.

$$AP_L = \frac{TP}{L}$$

## 3. Marginal Rate of Technical Substitution (MRTS)

MRTS is a measure of the degree of substitutability of factors of the production. How technically one factor can substitute for another so as to keep output level unchanged.

$$\text{MRTS}_{L,K} = \frac{\text{MP}_L}{\text{MP}_K}$$

## 4. Elasticity of substitution

This indicates the ease with which one factor, say labour can be substituted for other factor, say capital or any other factor.

$$\sigma = \frac{\text{percentage change in } K/L}{\text{percentage change in MRTS}}$$

$$\sigma = \frac{d(K/L)/(K/L)}{d(\text{MRTS})/(\text{MRTS})}$$

## 5. Factor intensity

This is the factor intensity property of technology. It shows the quantity of one factor (say capital) in relation to other factor (say labour). Capital-labour ratio is used to measure the factor intensity.

## 6. Efficiency of production

The efficiency of production refers to the entrepreneurial and organizational aspects of production. Two firms with identical factor inputs may have different levels of output due to the differences in their entrepreneurial and organizational efficiency. Most of the production functions include a separate parameter to measure the efficiency of production which have own influence on the output.

## 7. Returns to scale

This indicates proportionate change in output due to the equi-proportionate change in all inputs. There may be increasing, decreasing or constant returns to scale. Returns to scale is relevant to the long-run production.

# Neo-classical properties of a production function

1. For a continuous, single valued production function, the marginal products of the factors must be positive.

$$\frac{\partial q}{\partial x_i} \geq 0, i = 1, \ldots\ldots n$$

2. For some ranges of the factors, their marginal products must be eventually declined with increase in the quantity of factors.

$$\frac{\partial^2 q}{\partial x_i^2} < 0, \quad i = 1, \ldots\ldots\ldots \text{n}$$

3. When the quantity of a factor increases, the level of output must reach to a finite limit.

$$L \xrightarrow{\text{limit}} \infty, V = M_1 \qquad \text{K} \xrightarrow{\text{limit}} \infty, V = M_2$$

$M_1$ and $M_2$ are finite positive constants.

- All these characteristics of the production functions, including neo-classical properties are important not only in the theoretical point of view but also from the decision-making process in practice.

# Homogeneous Production Functions

The production function which reveals either increasing or decreasing or constant returns to scale throughout is called homogeneous PF.

The production functions which reflect all three types of returns to scale are called non-homogeneous PF.

The laws of returns to scale can be explained in a better way by using homogeneous PF.

The PF $q = f(x_1, x_2 \ldots . x_n)$ is said to be a homogeneous of degree $r$ if the following result hold for it.

$$f(kx_1, \ldots . kx_n) = k^r f(x_1, \ldots . x_n)$$

$$f(kx_1, \ldots . kx_n) = k^r q \qquad \ldots .. \quad (1)$$

Where $k$ shows the identical increase of all inputs and $r$ is a constant defined as the degree of homogeneity of PF.

Equation (1) shows that when all inputs increased by $k^{th}$ scale, output increases by $r^{th}$ scale.

If $r > 1$ increasing returns to scale

$r < 1$ decreasing returns to scale

$r = 1$ constant returns to scale

If we differentiate both sides of equation (1) with respect to $k$, we will have,

$$f_1 \frac{d(kx_1)}{dk} + f_2 \frac{d(kx_2)}{dk} .... + f_n \frac{d(kx_n)}{dk} = rk^{r-1}q$$

When $k$ =1, the above result becomes,

$$f_1 x_1 + f_2 x_2 + .... + f_n x_n = rq$$

If $q = f(x_1, x_2 .... x_n)$ is homogeneous of degree $r$, it is necessary to satisfy the above identity.

This is known as Euler's theorem for a homogeneous production function of degree r which has important application in economics.

eg: Consider the PF $q = AL^{\alpha}K^{\beta}$

If both inputs are increased by 10%, new output level:

$$q^* = A(1.10L)^{\alpha}(1.10K)^{\beta}$$

$$q^* = (1.10)^{\alpha+\beta}AL^{\alpha}K^{\beta}$$

$$q^* = (1.10)^{\alpha+\beta}q.$$

If $\alpha+\beta > 1$ increasing returns to scale

$\alpha+\beta < 1$ decreasing returns to scale

$\alpha+\beta = 1$ constant returns to scale

# Some Standard Production Functions

## The Cobb-Douglas Production Function

This is the simplest and most widely used production function in the economic theory and empirical studies. Originally it was developed by CW Cobb and PH Douglas (1928, ‘Theory of Production’. *Journal of Economic Review*. vol.18, p 225-250).

Original form was $V = AL^{\alpha} K^{\beta}$

V = flow of output
L = quantity of labor input
K= quantity of capital input
A= efficiency parameter

$\alpha$, $\beta$ = positive constants that show the output elasticities of L and K respectively. The values of these are determined by the available technology. $\alpha, \beta < 1$ and $\alpha+\beta =1$

$\alpha + \beta$ shows the degree of homogeneity and returns to scale.

The estimated function by Cobb and Douglas was,

$$V = 1.01 L^{0.75} K^{0.25}$$

Since $\alpha + \beta = 1$, C-D is a homogeneous production function that shows the constant returns to scale.

When L and K increase from a given proportion, output will be increased by the same proportion.

## C-D *pf* satisfy some of the Neo-classical properties:

1. For a continuous, single valued production function the marginal product of the factors must be positive.

$$MP_L = \frac{\partial V}{\partial L} = \alpha A L^{\alpha-1} K^{\beta} = \alpha \frac{V}{L}$$

$$= \alpha AP_L$$

$$MP_K = \frac{\partial V}{\partial K} = \beta A L^{\alpha} K^{\beta-1} = \beta \frac{V}{K}$$

$$= \beta AP_K$$

Since $\alpha, \beta < 1$ and $\alpha+\beta = 1$, both $MP_L$ and $MP_K$ are positive.

2. Marginal product of factors must be eventually declined.

$$\frac{\partial^2 V}{\partial L^2} = \frac{\alpha(\alpha - 1)}{L}\frac{V}{L} < 0$$

$$\frac{\partial^2 V}{\partial K^2} = \frac{\beta(\beta - 1)}{K}\frac{V}{K} < 0$$

$$\Theta \quad (\alpha - 1) < 0$$

3. C-D function does not satisfy this property.

## MRTS of C-D *pf*.

MRTS measures the degree of substitutability of factor inputs.

$$MRTS_{L,K} = \frac{\partial V/\partial L}{\partial V/\partial K} = \frac{\alpha(V/L)}{\beta(V/K)} = \frac{\alpha}{\beta}.\frac{K}{L}$$

*As a measure of the degree of substitutability of factor inputs MRTS has a serious defect: it depend on the units of measurement of the factors.*

## Elasticity of Substitution of C-D *pf*.

*Elasticity of substitution (σ) is a measure of the degree of substitutability among factors which is independent from the measurement units of factors*.

It can be defined as *the percentage change in the capital–labor ratio, divided by the percentage change in the rate of MRTS*.

$$\sigma = \frac{d(K/L)\big/(K/L)}{d(MRTS)\big/(MRTS)}$$

$$= \frac{d(K/L)/(K/L)}{d\left(\frac{\alpha}{\beta}\cdot\frac{K}{L}\right)\Big/\left(\frac{\alpha}{\beta}\cdot\frac{K}{L}\right)} = 1$$

Elasticity of substitution for L and K derived from the C-D *pf* is equal to 1. This means that L is perfectly substituted for K. This is one of the serious drawbacks of the C-D function because practically L and K are not perfect substitutes.

## Factor intensity

In the C-D function, factor intensity is measured by the α/β ratio. When the ratio is higher, production technology is more labor intensive and when it is lower more capital intensive.

## Efficiency of Production

- In the C-D function, efficiency of the production process i.e. efficiency of organization of the factors of production is measured by the coefficient *A*.

- The two firms, which possess identical factor inputs may have different output levels due to the difference of the efficiency of the production process. More efficient firms will have a large *A* than the less efficient firms.

## Elasticity of Production

Elasticity of production measures the percentage change of total output as a result of percentage change of one factor while other factors remain constant.

$$E_{PL} = \frac{Percentage of\ change of\ output}{Percentage of\ change of\ labor\, input}$$

$$E_{PL} = \frac{(\partial V / V) \times 100}{(\partial L / L) \times 100}$$

$$= \frac{\partial \mathrm{V}}{\partial \mathrm{L}} \times \frac{L}{V} = \frac{MP_L}{AP_L}$$

In the C-D function

$$MP_L = \alpha \frac{V}{L} \quad \text{and} \quad AP_L = \frac{V}{L}$$

$$\therefore \; E_{PL} = \alpha$$

Similarly

$$E_{PK} = \beta$$

This implies that in the C-D function, elasticity of production of inputs is given directly by the parameters of the function.

## Strengths/Merits

*C-D function is widely used in theoretical and empirical analysis.*

- The simplest form of the function and simplicity of estimation

It is an intrinsically linear function, although it appears as a non-linear function. By log transformation it can be transformed into a linear function.

- Thus, its parameters can be estimated easily by using linear techniques such as OLS.

$$\ln V = \ln A + \alpha \ln L + \beta \ln K$$

- Although original function restricted only to the two factors, it can be expanded to any number of factors.

$$V = \beta_0 X_1^{\beta_1} X_2^{\beta_2} \ldots . X_n^{\beta_n}$$

$$\ln V = \ln \beta_0 + \beta_1 \ln X_1 + \beta_2 \ln X_2 + \ldots . + \beta_n \ln X_n$$

- It satisfies the some Neo-classical properties,
- It provides measures on several concepts which are useful tools in all fields of Economics.

## Weaknesses

- Degree of substitutability among factors is not realistic. $\sigma = 1$
- All the inputs include in the function must be positive. If one of the factor inputs of the function is zero, total output will be zero.
- It shows the constant returns to scale. $\alpha + \beta = 1$
- Assumption of constant technology: technology is a dynamic concept.

## Constant Elasticity of Substitution (CES) Production Function

One of the drawbacks of C-D *pf* was that the perfect substitutability between labor and capital inputs. σ = 1.

The CES *pf* specifies constant elasticity of substitution between capital and labor but not necessarily at unit level.

The CES pf has been derived independently by two groups of economists:

- Arrow, Chenery, Minhas and Solow (1961) – ACMS version
- Brown and deCani (1963) – Brown's version

### Brown's version of CES *pf*

$$q = \gamma\left[\delta K^{-\rho} + (1-\delta)L^{-\rho}\right]^{-\nu/\rho}$$

Where,

q = level of output

K = level of capital input

L = level of labor input

γ = parameter shows the efficiency of technology

δ = input intensity parameter

ν = represent the degree of homogeneity of the function and therefore the degree of returns to scale

ρ = substitution parameter. This is a transformation of

$$\sigma = \frac{1}{1+\rho}$$

The range of ρ is $\infty \leq \rho \leq -1$ for which $\sigma \geq 0$

## *ACMS* version of CEF *pf*.

This function shows the constant returns to scale i.e. $v = 1$

$$q = \gamma\left[\delta K^{-\rho} + (1-\delta)L^{-\rho}\right]^{-1/\rho}$$

Marginal products of L and K

$$MP_L = \frac{\partial q}{\partial L} = (1-\delta)\gamma^{-\rho}\left(\frac{q}{L}\right)^{1+\rho} \qquad MP_K = \delta\gamma^{-\rho}\left(\frac{q}{K}\right)^{1+\rho}$$

Marginal productivities are depend on the input intensity (δ), efficiency of technology ($\gamma$) and the elasticity of substitution [(1+ρ)=1/σ].

- $MP_L$ and $MP_K$ are positive for non-zero values of the inputs

- By taking second order derivatives ($\partial^2 q/\partial K^2$) and ($\partial^2 q/\partial L^2$) we can show marginal products of L and K are decline as L and K increase.

The CES *pf* has been generalized by number of economists mainly by Lu and Fletcher. The new function is a special case of CES which is called *Variable Elasticity of Substitution* (VES) *pf*.

$$q = \gamma\left[\delta K^{-\rho} + (1-\delta)\eta(K/L)^{c(1+\rho)}L^{-\rho}\right]^{-1/\rho}$$

Although CES as well as VES *pfs* are theoretically elegant, the estimation process is very complicated. Therefore they are hardly used in empirical studies.

# Decision periods of production analysis

There are two different decision periods over which we examine the production theory namely *short-run* and *long-run.*

## Short-run production

The period of time in which quantities of one or more factors of production cannot be changed.

This implies that in the short-run there are two categories of inputs:

- Fixed inputs
- Variable inputs,

As such there are two types of costs: *fixed cost* and *variable cost*.

Agricultural production is an example for the short-run production- land is the fixed factor and labor is the variable factor.

## Short-run production function

$$q = f(L, \bar{K})$$

q = quantity of output

L = quantity of labour input – variable factor

$\bar{K}$ = quantity of capital input – fixed factor

## Long-run production

- The period of time in which quantities of all factors of production can be varied.
- In the long-run production, all factors are variable factors. However, the production technology further remains unchanged.

## Long-run production function

$q = f(x_1, x_2, \ldots\ldots, x_n)$ q is output and all $x_i$'s are variable factors.

# Short-run Behavior of Production

In the short-run, firms carryout production by increasing the amount of variable factor, keeping other factors unchanged. This production process is subjected to the ***Law of Variable Proportion* or *the Law of Diminishing Marginal Productivity.***

## The Law of Variable Proportion

The law of variable proportion refers to the behavior of output as the quantity of one factor increase, keeping the quantity of other factor/s constant.

When the amount of the variable factor increases while keeping amount of fixed factor/s constant, the amount added to the total product (TP) by an additional unit of the variable factor, i.e. *marginal product* of the variable factor, will eventually decline.

If the form of the production function is,

$$q = f(L, \overline{K})$$

According to the law, when L increase by equal increments while K held constant, the total output will increase first at an increasing rate, then at decreasing rate. Beyond a specific point of input, TP will decline. Following the law, marginal productivity of labour ($MP_L$) will decline eventually.

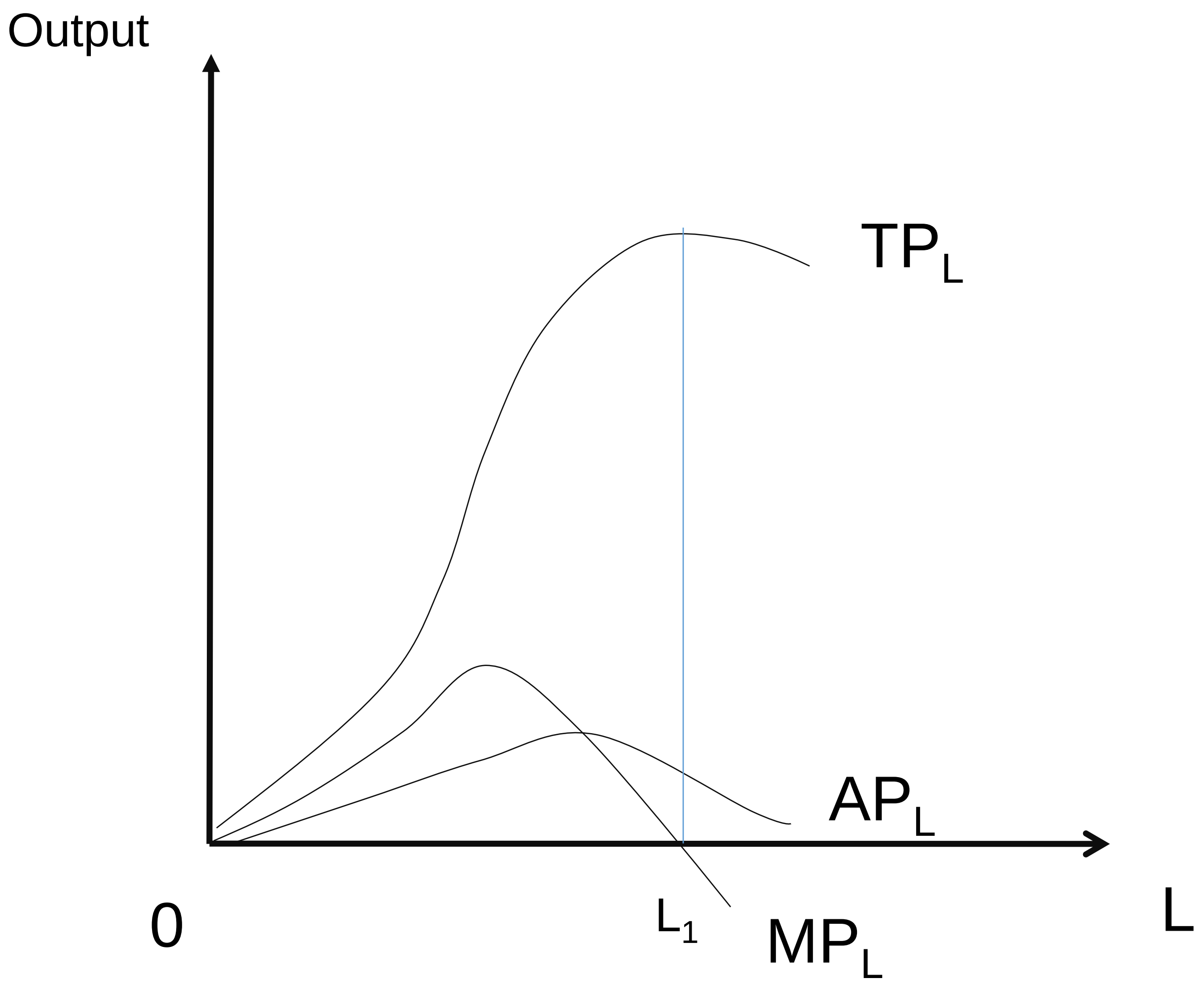
This implies that

$$\partial^2 q / \partial L^2 < 0$$

# The Law of Variable Proportion

| K | L | TP | AP | MP | Marginal Productivity |
|---|---|---|---|---|---|
| 25 | 1 | 240 | 240.0 | 240 | |
| 25 | 2 | 720 | 360.0 | 480 | |
| 25 | 3 | 1380 | 460.0 | 660 | Increasing |
| 25 | 4 | 2160 | 540.0 | 780 | |
| 25 | 5 | 3000 | 600.0 | 840 | |
| 25 | 6 | 3840 | 640.0 | 840 | |
| 25 | 7 | 4620 | 660.0 | 780 | |
| 25 | 8 | 5280 | 660.0 | 660 | |
| 25 | 9 | 5760 | 640.0 | 480 | Decreasing |
| 25 | 10 | 6000 | 600.0 | 240 | |
| 25 | 11 | 6000 | 545.5 | 0 | |
| 25 | 12 | 5940 | 495.0 | -60 | |
| 25 | 13 | 5520 | 424.6 | -420 | Negative |

Graphically, the relationship between the TP, AP, MP, and quantity of variable factor can be described as:



## Assumptions of the Law

The law is valid when the following assumptions are satisfied:

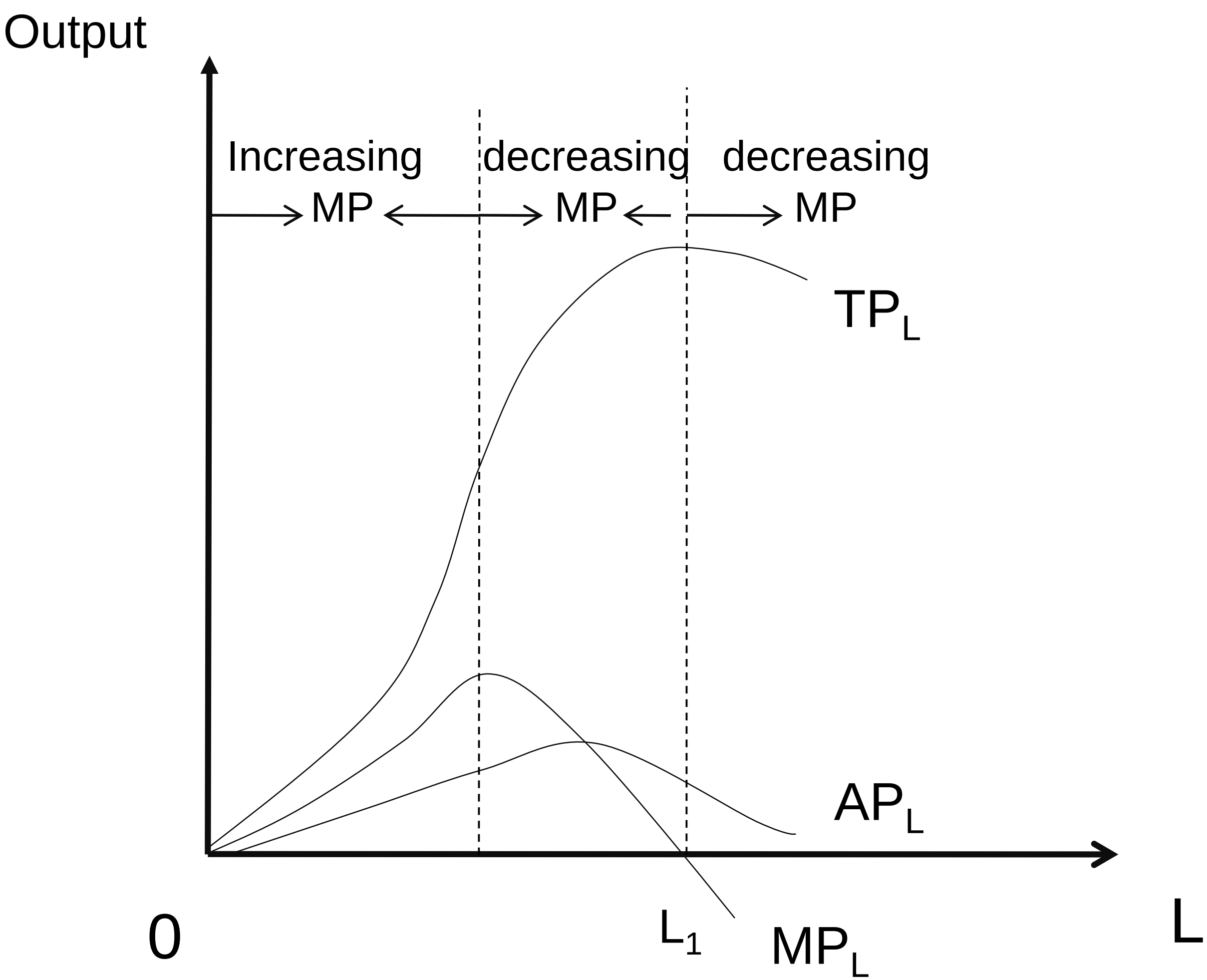
- There is at least one fixed factor with which the variable factor cooperates in production.
- All units of the variable factor are homogeneous.
- Factors can be combined in varying proportion to produce the product.
- State of technology is assumed to be given/constant.

If the quantity of $\overline{K}$ increases, the TP curve will shift upward, but the shape of TP curve will not change since the technology is unchanged.

However, technology advances over time due to the inventions and other improvements of technology. With the advancement of technology, total output (TP) curve shift (together with AP and MP) upward indicating more output can be produced with the same inputs.

# Three types of Marginal Productivity

By observing the behavior of the product curves, we can identify 3 types of marginal productivities i.e. increasing MP, decreasing MP and Negative MP.



**Increasing MP-** Every additional unit of L increases TP, AP and MP. This is shown in the graph by the increasing slope of TP curve.

**Decreasing MP –** Every additional unit of L, decreases MP though TP still increases. This is shown in the graph by the negative slope of MP and decreasing positive slope of TP curve.

**Negative MP –** Every additional unit of L decreases MP as well as TP. This is shown in the graph by the negative slope of MP as well as TP curves. Numerical value of MP is negative.

In a classical production function model, these 3 types of MPs are included.

**Three Stages of Production**

Conventionally, short-run production process is divided into three phases or stages based on the relationship between TP, AP and MP of the variable factor.

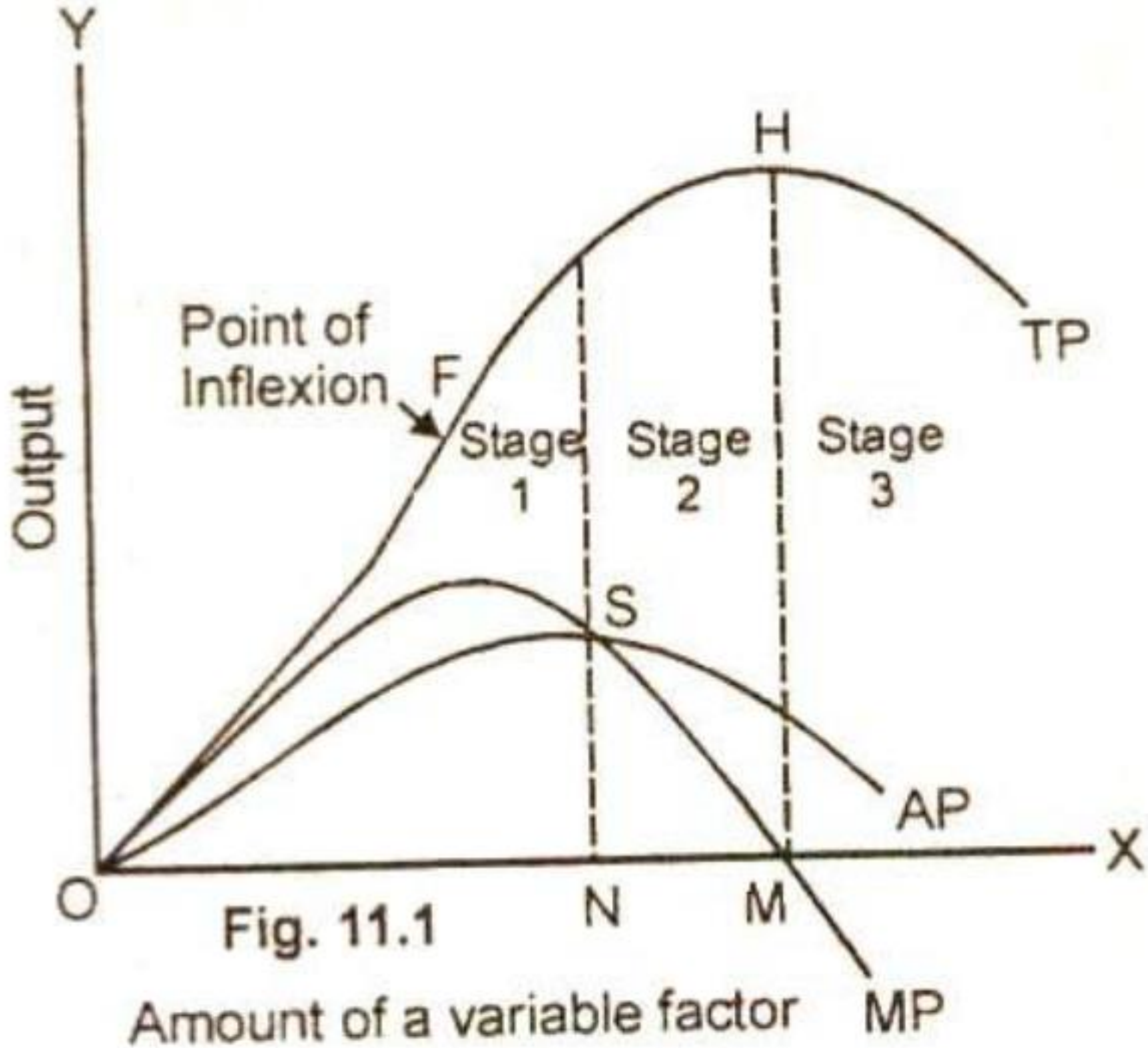



Fig. 11.1

| Stages | Total Product | MP | AP |
|---|---|---|---|
| Stage I MP>AP | Initially it increases at an increasing rate and later at decreasing rate. | Initially increases and reaches the maximum point. Then starts decline. | Increases and reaches its maximum point. At the end point of the stage AP=MP |
| Stage II MP<AP | Increases at diminishing rate and reaches its maximum point. | Decreases and become zero at the point where TP is maximum- at point M | After reaching its maximum point, begins to decrease. |
| Stage III MP<0 | Begins to fall | Becomes negative | Continuously decline but remains positive. |

## Stage of Operation

A Rational producer will not operate in,

- Stage III as MP of variable factor is negative
- Stage I as factors of production are under utilized.

A rational producer will always produce in,

- stage II where both the MP and AP of the variable factor is positive though it is diminishing.

- At which particular point in this stage a producer decides to produce depends upon the prices of factors.

- However, if the output maximization is the goal and there is no constraints, then the end point of stage II, i.e. the point where the MP curve cuts the horizontal axis ($MP_L = 0$) would be the most desirable input combination.