Mathematics for Economics
MA/MSSc in Economics-2020/2021

Prof. W. M. Semasinghe
Department of Economics

MATHEMATICS AND STATISTICS

LERNING OUTCOMES:

By the end of this course unit students will be able to demonstrate skills in mathematical and statistical methods that are highly useful in analyzing problems related to economic theory and practice and understand the uses of basic descriptive and inferential statistics in economic analysis.

COURSE CONTENTS: This course unit consists of two parts:

Part I: Mathematics - Functions and their applications in economics, Calculus and its applications in economics: Differentiation, Partial differentiation, Integration; Matrices; Maxima and Minima; Constrained Optimization with economic applications, Linear programming.

Part II: Statistics - Probability and probability Theory; continuous and discrete variables, continuous and discrete variables distributions, Joint distributions; Moment generating functions; Hypothesis testing and confidence intervals.

Functions

❖ *A function is a devise that explain the relationship between variables.*

If the value of variable ***y*** depends on the values of another variable say ***x***, y is said to be a function of x and, denoted by,

$$y = f(x)$$

Values of x is called *input numbers*

Values of y is called *output numbers*

For each value of x, y can take only one value

❖ *A function is a rule that assigns to each input number exactly one output number.*

The variable that represents input numbers is called independent variable

The variable that represents output numbers is called dependent variable

$$y = f(x)$$

Dependent variable

Independent variable

- The set of all input numbers to which the rule applies is called the ***domain*** of the function.

- The set of all output numbers is called the ***range***.

Unless otherwise stated, the domain of a function consists of all real numbers (?) for which the function is defined.

eg:- $q = f(p)$ $s = f(y_d)$

$p = f(q)$ $c = f(y_d)$

$TC = f(q)$ $TR = f(q)$

To denote a function the symbols F, G, g, h, and the Greek letters such as θ, φ, Φ, ψ, Ψ etc. are also used.

If y and z depend on x, then $y = f(x)$ and $z = f(x)$
We can also write

$$y = y(x)$$
$$z = z(x)$$

e.g. (1) $y = 18x - 3x^2$

If the input (domain) x is any real number, the output (range) y is also a real number, R.

e.g. (2) $$y = \frac{2x+3}{x-1}$$

The output ***y*** is a real number if the input ***x*** is any real number other than 1. The domain is any real number R – {1}

e.g. (3) Find the domain of $y = \dfrac{6}{x(x+9)}$

Denominator cannot be zero. $x \neq 0$ and $x \neq -9$

The domain is any real number but not 0 and -9

Identify the range,

1). $y = 2x$,

when domain (x) is any real number range (y) is also a real number.

2). $y = x^2$,

when domain (x) is any real number range $y \geq 0$

3). $y = -5x$, $(-1 \leq x \leq 2)$

The range is $-10 \leq y \leq 5$

Most variables in economic models are by their nature restricted to non-negative real numbers. Thus, their domains are also restricted.

eg. The total cost of a firm per day is a function of its daily output Q:
C = 150 + 7q. The firm's capacity has limited to 100 units of output per day.

What are the domain and rang of the cost function?

$$Domain = \{q \mid 0 \leq q \leq 100\}$$

$$Range = \{C \mid 150 \leq C \leq 850\}$$

If ‘a’ is any particular value of the function of x, the value of the function $f(x)$ for x = a is denoted by f(a).

eg. 1 $y = f(x) = \dfrac{x}{7x + 1}$

$f(a) = a/(7a + 1)$

$f(4) = 4/\ 29$

Given,

$f(x) = x^2 + 5x – 6$ Find, $f(3)$ and $f(-4)$

$f(x) = \dfrac{4x^2 - 9x + 17}{x + 7}$ Find, $f(5)$ and $f(-2)$

$f(x) = \dfrac{x^2 - 11}{x + 4}$ Find, f(3a) and f(a - 4)

$f(x) = 1 – 2x$; Find $f(3)$, $f(-x)$, and $f(x+h)$

Types of Functions

- Constant function: A function whose range consists of only one element.

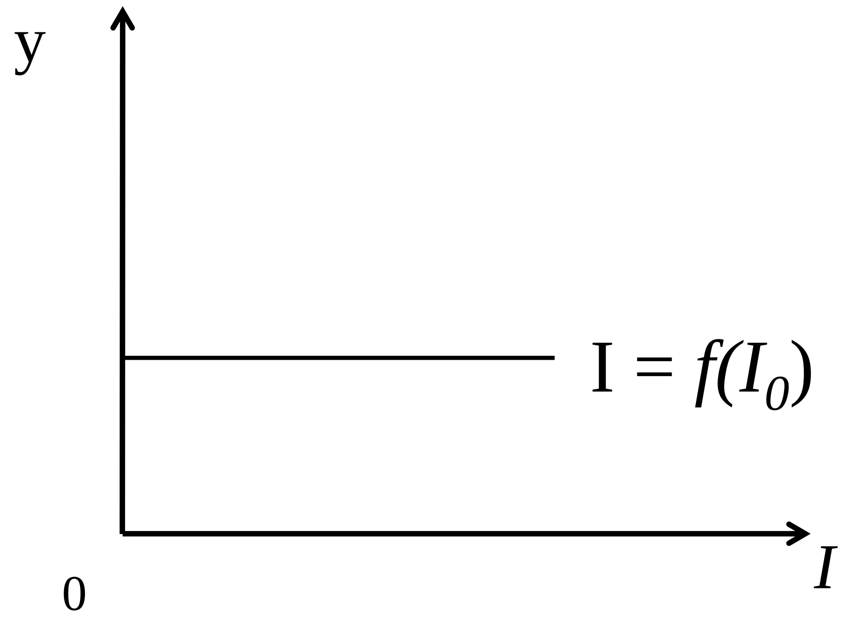
eg. $y = 7$

$f(x) = 10$

The value of the function does not change whatever the value of x.

In a coordinate plane, such a function will appears as a horizontal straight line.

eg. In national income models, when investment (I) is exogenously determined, the investment function appears as a horizontal straight line.



Polynomial Functions

The general form of a Polynomial Functions of Degree n

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_{n-1}x^{n-1} + a_nx^n$$

n is a positive integer and a_0 …..a_n are constants. $a_n \neq 0$

e.g. $f(x) = 8x^6 + 3x^4 - x^3 + 5x^2 + 2x + 3$ (polynomial of degree 6)

$F(x) = x^8 + 2x^5 + 3x^4 + 7x^2 + 6x - 5$ (polynomial of degree 8)

Depending on the value of the integer n there are several subclasses of polynomial functions.

when

n = 0 $y = a_0$ Constant function

n = 1 $y = a_0 + a_1x$ Linear function (polynomial of degree 1)

n = 2 $y = a_0 + a_1x + a_2x^2$ Quadratic function

n = 3 $y = a_0 + a_1x + a_2x^2 + a_3x^3$ Cubic function

Rational Functions

A function of the form $$f(x) = \frac{g(x)}{h(x)}$$

$g(x)$ and $h(x)$ are polynomials and $h(x) \neq 0$

$f(x)$ is expressed as a ratio of two polynomial functions of variable x.

$$f(x) = \frac{3x^2+5)}{(2x-1)}$$

Composite Function or Function of a Function

If $y = g(u)$ and $u = f(x)$, $y = g[f(x)]$

eg. (1). If $y = u^2 + 3$ and $u = 2x + 1$ then,

$$y = (2x+1)^2 + 3$$

eg. (2). If $y = x^3 - 3x + 5$ and $x = \frac{1}{2}\sqrt{t} + 3$ then,

$$y = ?$$

$$y = (\tfrac{1}{2}\sqrt{t}+3)^3 - 3(\tfrac{1}{2}\sqrt{t}+3) + 5$$

Non-algebraic Function

The function which the independent variable appears as the exponent is called non-algebraic function.

$$y = e^t$$

$$y = b^x$$

Logarithmic functions are also non- algebraic functions.

$$y = \log_b x$$

$$y = \ln x$$

Functions of two or more independent variable

i. $z = f(x, y)$ ii. $z = a_0 + a_1x + a_2x^2 + b_1y + b_2y^2$

iii. $z = ax + by$ iv. $y = a_1x_1 + a_2x_2 + a_3x_3 + \ldots a_nx_n$

v. $V = AL^{\alpha}K^{\beta}$

The arithmetic of functions

Functions can be added, subtracted, multiplied and divided to form new functions. Given the two functions f and g of x:

i) Their sum, denoted by $f + g$ is the function defined by $(f + g)(x) = f(x) + g(x)$

ii) Their difference, denoted by $f - g$ is the function defined by $(f - g)(x) = f(x) - g(x)$

iii) Their product, denoted by $f . g$ is the function defined by $(f . g)(x) = f(x) . g(x)$

iv) Their quotient, denoted by f / g, is the function defined by $(f / g)(x) = \frac{f(x)}{g(x)}$

e.g. Given that the total cost and revenue of manufacturing and selling q units of a certain commodity is C(q) and R(q) respectively. Therefore, the profit function of that commodity is defined as $\pi(x) = R(q) – C(q)$

Ex. Given that f and g are functions defined by $f(x) = x^4 + x$ and $g(x) = x + 1$, find: $f+g$, $f - g$, $f . g$ and f/g

Differentiation

The process of finding the derivative (deferential coefficient) of a function is called *differentiation*. It stands for finding the rate of change of one variable, say y, with respect to the rate of change of another variable, say x.

For a function $y = f(x)$ the differential coefficient of y with respect to x is defined as

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$y = f(x) \qquad (1)$$

Assume, when variable x is increased by Δx, variable y will also be increased by Δy,

$$y + \Delta y = f(x + \Delta x) \qquad (2)$$

(2) - (1) $$\Delta y = f(x + \Delta x) - f(x) \qquad (3)$$

(3) ÷ Δx $$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The limit of $\Delta y/\Delta x$ when $\Delta x \xrightarrow{\lim} 0$, is defined as the *differential coefficient* of $f(x)$

- dy/dx is used to denote the limiting value $\Delta y/\Delta x$

Rules of Differentiation

When u, v and y are the functions of x and k is a constant,

(1) Constant function rule

$$y = k$$

$$\frac{dy}{dx} = \frac{d(k)}{dx} = f'(x) = 0$$

$$y = 10$$

$$\frac{dy}{dx} = \frac{d(10)}{dx} = f'(x) = 0$$

(2) Linear function rule

$$y = f(x) = kx + 2$$

$$\frac{dy}{dx} = \frac{d(kx+2)}{dx} = k$$

$$y = 9x + 6$$

$$\frac{dy}{dx} = \frac{d(9x+6)}{dx} = 9$$

(3) Power function rule

$y = f(x) = x^{n}$

$$\frac{dy}{dx} = \frac{d(x^{n})}{dx} = nx^{n-1}$$

e.g. $y = x^{6}$

$$\frac{dy}{dx} = \frac{d(x^{6})}{dx} = 6x^{5}$$

Ex. (1) $y = x^{5/2}$

Ex. (2) $y = x^{2/3}$

Ex. (3) $y = x^{-3}$

Ex. (4) $y = x^{-5/2}$

(4). Generalized power function rule

$y = kx^n$

$$\frac{dy}{dx} = \frac{d(kx^n)}{dx} nkx^{n-1}$$

Ex (1) $y = 2x^4$

Ex. (2) $y = ax^3$

Ex. (3) $y = ax^{n+1}$

Ex. (4) $y = 1/x$

Ex. (5) $y = f(t) = 8t^6$

Rules of differentiation involving two or more functions of the same variable

(5). Sum-difference rule

$$y = [f(x) \pm g(x)]$$

$$\frac{dy}{dx} = \frac{d[f(x)]}{dx} \pm \frac{d[g(x)]}{dx}$$

Ex. (1) $y = 2x^3 + 4x^2 - 8x + 18$

Ex. (2) $y = 2x^{1/2} + 6x^{1/3} - 2x^{2/5} + 18$

(6). Chain Rule or Function of a function rule

If $y = f(u)$ and $u = g(x)$, then $y = f[g(x)]$ is a function of x.

If y is a differentiable function of u and u is a differentiable function of x, then $y = f[g(x)]$ is a differentiable function of x.

To obtain dy/dx we can use two methods:

1). Express y explicitly in terms of x and differentiate

e.g. $y = u^2+3$ and $u = 2x+1$, then $y = (2x+1)^2 + 3 = 4x^2+4x+1$

$$\therefore \quad \frac{dy}{dx} = 8x + 4$$

2). Differentiate each function separately with respect to independent variable and use the *chain rule*

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{dy}{du} \times \frac{du}{dx}$$

Derivative of the above example is,

$$\frac{\mathrm{dy}}{\mathrm{du}} = 2u \qquad \text{and} \qquad \frac{du}{dx} = 2$$

$$\therefore \qquad \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{dy}{du} \times \frac{du}{dx} = 4u = 4(2x+1) = 8x+4$$

e.g. $y = (ax+b)^n$

If we defined $(ax + b) = u$ then $y = u^n$

$$\frac{\mathrm{dy}}{\mathrm{du}} = nu^{n-1} \qquad \text{and} \qquad \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{d(ax+b)}{dx} = a$$

$$\therefore \qquad \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{dy}{du} \times \frac{du}{dx} = n(ax+b)^{n-1} \times a = an(ax+b)^{n-1}$$

We can express this procedure as a rule as follows:

When $y = [f(x)]^n$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = n[f(x)]^{n-1} f'(x)$$

Ex. (1) $y = 3(2x + 5)^2$

Assume that $(2x + 5) = u$, then $y = 3u^2$

Ex. (2) $y = (3 - 4x^2)^{3/2}$

If $(3 - 4x^2) = u$, $y = (u^{3/2})$

(7). Product rule

$$y = [f(x).g(x)]$$

$$\frac{dy}{dx} = f(x)\frac{d[dg(x)]}{dx} = g(x)\frac{d[f(x)]}{dx}$$

$$f(x) = u \text{ and } g(x) = v$$

$$y = uv$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

e.g. $y = (2x^3 - 3x)(x^2 + 5)$

If $(2x^3 - 3x) = u$ and $(x^2 + 5) = v$

$$\frac{dy}{dx} = (2x^3 - 3x)\frac{d(x^2+5)}{dx} + (x^2+5)\frac{d(2x^3-3x)}{dx}$$

$$= (2x^3 - 3x)(2x) + (x^2 + 5)(6x^2 - 3)$$

If u, v and w are functions of x,

$$y = f(u,v,w)$$

$$\frac{dy}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

Ex. $y = x^2 (x^2 + 1) (x + 2)$

(8). Quotient rule

$f(x) = u \qquad g(x) = v$

$$y = \frac{f(x)}{g(x)}$$

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{g(x)\frac{d[f(x)]}{dx} - f(x)\frac{d[g(x)]}{dx}}{[g(x)]^2}$$

$$\frac{dy}{dx} = \frac{v\frac{d(u)}{dx} - u\frac{d(v)}{dx}}{(v)^2}$$

Ex. (1) $y = \dfrac{(2x-3)}{(x+1)}$

When $y = 1/f(x)$

$$\frac{dy}{dx} = \frac{-f'(x)}{[f(x)]^2}$$

Prove this!

e.g. $y = 1/(x^2+1)$

$$\frac{dy}{dx} = \frac{-2x}{(x^2+1)^2}$$

(10). Log-function rule

When a variable is expressed as a function of logarithm of another variable, the function is referred to as a *logarithmic function* or *log function*.

$$y = log_b\, x \qquad y = log_e\, x \;\; (= ln\, x)$$

Which differ from each other only in regards to the base of logarithm.

In calculus *e* takes as the base of logarithm.

Logarithms on base *e* is called *natural logarithm* and denoted as $\log_e$ or ln.

e.g. $y = log_e\, 2x$ or $y = ln\, 2x$

1. $y = \ln x$

$$\frac{dy}{dx} = \frac{d(\ln x)}{dx} = \frac{1}{x}$$

2. $y = k(\ln x)$

$$\frac{dy}{dx} = k\frac{d(\ln x)}{dx} = k\frac{1}{x} = \frac{k}{x}$$

3. If $y = \ln u$ and $u = f(x)$

$$\frac{dy}{dx} = \frac{1}{u} \times \frac{du}{dx}$$

4. $y = \ln f(x)$

$$\frac{dy}{dx} = \frac{1}{f(x)} f'(x)$$

Ex. (1) $y = \ln (3x^2 + 5)$

Ex. (2) $y = x^3 \ln x^2$

Log of a product

$\ln(uv) = \ln u + \ln v$

$\text{Ln}[(x^3+2)(x^2+3)] = ?$

Log of a quotient

$\ln(u/v) = \ln u - \ln v$

$\ln [x^4/(3x - 4)^2] = \ln x^4 – \ln (3x-4)^2 = 4 \ln x – 2 \ln(3x-4)$

$dy/dx = ?$

(11). Exponential Function Rule

Power (index) of the independent variable in a polynomial function is called *exponent*.

e.g. $y = 2x^3 + 4x^2 + 5$

The functions which appears independent variable as exponent is called *Exponential function*.

e.g. $y = f(x) = b^x$ $(b>1)$

where, y and x are dependent and independent variables respectively and b is the *base* of the exponent.

In calculus, an irrational number e (e = 2.71828….) takes as the base of the exponent.

eg. $y = e^x$ $y = e^{3x}$ $y = Ae^{rx}$

$$y = e^x$$

$$\frac{dy}{dx} = e^x$$

e.g. (1) If $y = e^u$ and $u = f(x)$

$$\frac{dy}{dx} = e^u \frac{du}{dx} = e^u f'(x)$$

e.g. (2) $y = e^{(3x + 2)}$

$$\frac{dy}{dx} = 3e^{(3x + 2)}$$

Derivatives of Higher Order

When derivative of a function is differentiable, it may have a derivative as well.

If y = f(*x*), then d*y*/d*x* or $f'(x)$ is called 1st derivative of the function.

The derivative of d*y*/d*x* or $f'(x)$ is denoted by d^2y/dx^2 or $f''(x)$ and is called the second derivative.

If $y = f(x)$

1st derivative: $$f'(x) = \frac{d(y)}{dx} = \frac{dy}{dx}$$

3rd derivative: $$f'''(x) = \frac{d(d^2y/dx^2)}{dx} = \frac{d^3y}{dx^3}$$

2nd derivative: $$f''(x) = \frac{d(dy/dx)}{dx} = \frac{d^2y}{dx^2}$$

nth derivative: $$f^n(x) = \frac{d(d^{n-1}y/dx^{n-1})}{dx} = \frac{d^ny}{dx^n}$$

eg. $y = x^4 + 3x^2 - 2x + 7$

1st derivative: $$f'(x) = \frac{dy}{dx} = 4x^3 + 6x - 2$$

2nd derivative: $$f''(x) = \frac{d^2y}{dx^2} = 12x^2 + 6$$

3rd derivative: $$f'''(x) = \frac{d^3y}{dx^3} = 24x$$

Find

i. $y = x\sqrt{1-2x}$ 2nd

iii. $y = 2x^4 + 6x^3 - 12x^2 + 6x - 2$ 3rd

ii. $y = \dfrac{3x-1}{x+2}$ 3rd

iv. $y = x^3 + e^x$ 4th

Partial differentiation

So far we dealt with functions of a single independent variable x, which were generally expressed as $y = f(x)$. However, there are many situations in which we must consider more than one independent variables.

A function of two independent variables x and y can be expressed as,

$$z = f(x, y).$$

z is the dependent variable.

In general, if y is a function of n independent variables x_1, x_2, ...x_n, it can be written as

$$y = f(x_1, x_2, \ldots., x_n)$$

Given a multivariable function y = $f(x_1, x_2, \ldots., x_n)$, the *partial derivative* measures the rate of change of the dependent variable (y) as a result of change of one of the independent variables, while the other independent variables keep constant.

If z = $f(x, y)$, the partial derivative of ***z*** with respect to *x* is obtained by treating ***y*** as a constant and applying ordinary rules of differentiation.

The partial derivative of ***z*** with respect to ***x*** is symbolized by $\boldsymbol{\partial z/\partial x}$ or f_x.

Similarly, the partial derivative of ***z*** with respect to ***y*** is symbolized by $\boldsymbol{\partial z/\partial y}$ or $\boldsymbol{f_y}$.

e.g. 1 $y = f(x_1, x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$

$$\frac{\partial y}{\partial x_1} = f_1 = 6x_1 + x_2 \qquad \frac{\partial y}{\partial x_2} = f_2 = x_1 + 8x_2$$

e.g. 2. $z = f(x, y) = (x^2 - 7y)(x - 2)$

$$\frac{\partial z}{\partial x} = f_x = (x^2 - 7y)\frac{d(x-2)}{dx} + (x-2)\frac{d(x^2 - 7y)}{dx}$$

$$= (x^2 - 7y) + (2x^2 - 4x)$$

$$\frac{\partial z}{\partial y} = f_y = (x^2 - 7y)\frac{d(x-2)}{dy} + (x-2)\frac{d(x^2 - 7y)}{dy}$$

$$= (-7x + 14)$$

Ex. (1). If $f(x,y) = (2x - 3y)/(x + y)$ Find f_x and f_y

Ex. (2). Find f_x and f_y $z = f(x, y) = (x^2 - 7y)(x - 2)$

Ex. (3). If $f(x,y) = (2x - 3y)/(x + y)$, find f_x and f_y

Exercise (1) Find the partial derivatives (f_x and f_y) of the following functions.

i. $z = \ln(x^2 + y^2)$

ii. $z = (x + y)\, e^{(x + y)}$

iii. $z = 3x^2y + 4xy^2 + 6xy$

iv. $z = 2x_1^3 - 11x_1^2 x_2 + 3x_2^2$

(2). The indifference curve of a consumer is given as

$$U = U(x_1, x_2) = (x_1 + 2)^2 (x_2 + 3)^2$$

Where, U is the total utility, x_1 and x_2 are the quantity consumed from the two goods X and Y.

i. Derive the marginal utility functions of the two goods.

ii. If consumer buys 3 items from each good, find the marginal utility of 1^{st} good.

(3). Derive the marginal productivity functions of labor (MP_L) and capital (MP_K) from the production function

$$V = AL^{\alpha}K^{\beta} .$$

Where, V is the output, L and K are labor and capital inputs respectively.

(4) The cost function of a firm is given by $C = 2x^2 + x - 5$. Find (i) the average cost (ii) the marginal cost, when $x = 4$

(5) The total revenue received from the sale of x units of a product is given by $R(x) = 12x + 2x^2 + 6$.

Find (i) the average revenue

(ii) the marginal revenue

(iii) marginal revenue at $x = 50$

(iv) the actual revenue from selling 51st item

(6) The demand function of a product for a manufacturer is $p(x) = ax + b$

He knows that he can sell 1250 units when the price is Rs.5 per unit and he can sell 1500 units at a price of Rs.4 per unit.

Find the total, average and marginal revenue functions.

Also find the price per unit when the marginal revenue is zero.

Partial Derivatives of Higher Order

❖ Higher partial derivatives are obtained in the same way as higher derivatives.

❖ For the function z = $f(x, y)$, there are 4 second order partial derivatives:

$$\frac{\partial^2 z}{\partial x^2} = f_{(x)x} = f_{xx} = \frac{\partial(\partial z / \partial x)}{\partial x}$$

$$\frac{\partial^2 z}{\partial y^2} = f_{(y)y} = f_{yy} = \frac{\partial(\partial z / \partial y)}{\partial y}$$

$$\left.\begin{aligned} \frac{\partial^2 z}{\partial y \partial x} &= f_{(y)x} = f_{yx} = \frac{\partial(\partial z / \partial x)}{\partial y} \\ \frac{\partial^2 z}{\partial x \partial y} &= f_{(x)y} = f_{xy} = \frac{\partial(\partial z / \partial y)}{\partial x} \end{aligned}\right\} \text{Cross partial derivatives}$$

e.g. $z = 4x^6 - 3x^2y^2 + 5y^4$

$f_x = 24x^5 - 6xy^2$

$f_y = -6x^2y + 20y^3$

$f_{xx} = 120x^4 - 6y^2$

$f_{yy} = -6x^2 - 60y^2$

$f_{yx} = -12xy$

$f_{xy} = -12xy$

$f_{xy} = f_{yx}$

Ex. $z = x^2e^{-y}$

(1). Find four second order partial derivatives of each of the following functions:

$$z = 7x \ln(1 + y)$$

$$z = (2x + 5y)(7x - 3y)$$

$$z = e^{4x - 7y}$$

(2). i. Find 1st and 2nd order partial derivative of $z = 7x \ln(1 + y)$

ii. Show that $f_{xy} = f_{yx}$

(3) Consider the production function, $V = 20L^{½}K^{½}$

Where L and K are labor and capital inputs are respectively. Examine whether the inputs show the diminishing marginal producvt.

Optimization

Optimization is the choice of the best among the available alternatives.

Optimization

Maximization

- Profit
- Utility
- Growth rate
- Sales
- Output etc.

Minimization

- Cost
- Risk etc.

$$y = f(x)$$

This is called objective function, if the aim is to find the value of x which optimizes the value of dependent variable y. It may be a maximization or a minimization.

x – choice variable

- decision variable

- policy variable

x indicates the value that it should take to maximize/minimize variable y.

Suppose a business firm seeks to maximize profit (π). Profit is maximized at the output (Q) which maximize the difference between total revenue (R) and total cost (C). Given state of technology and market demand, both R and C are functions of Q. There for π is also a function of Q.

$$\pi = R(Q) - C(Q)$$

π indicates the goal of the firm i.e. what should optimize. Thus, it is the objective function of the optimization problem. Q is the choice variable. The aim is to find the value of Q that maximize the π.

Determination of maxima and minima of a function

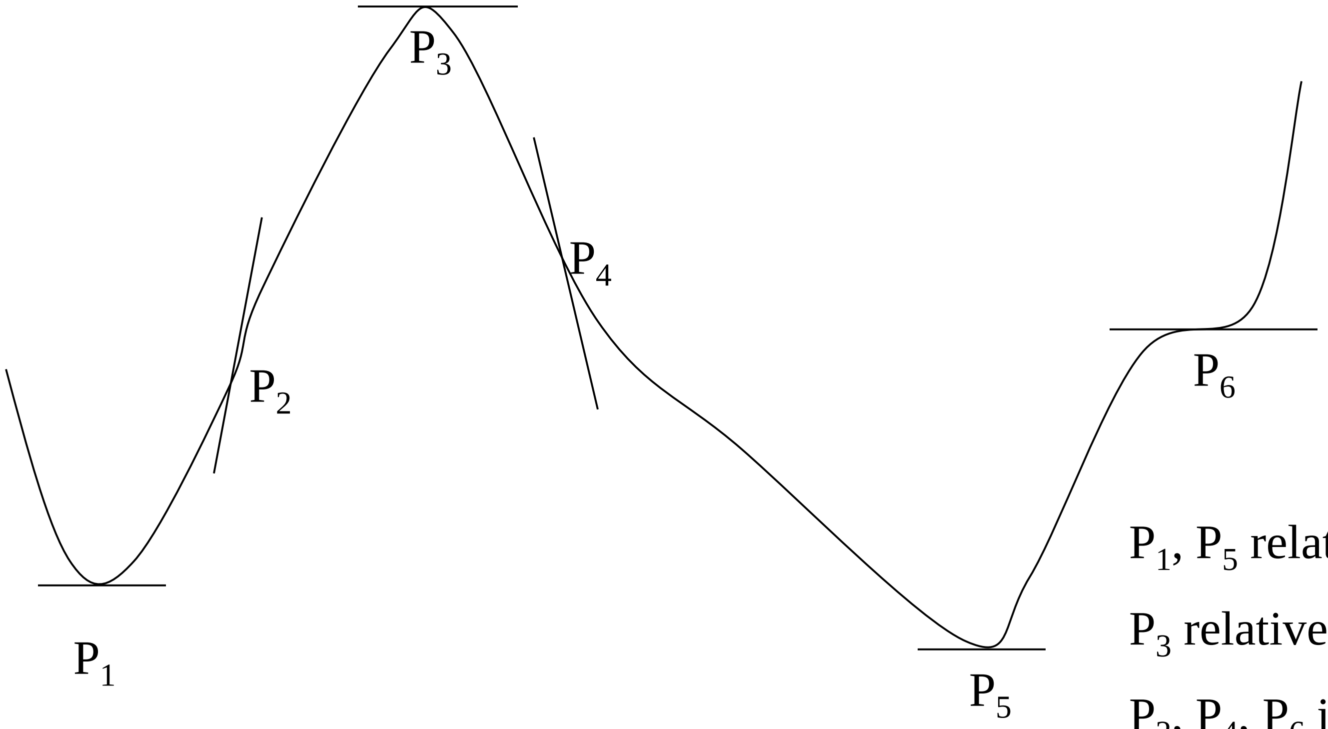
- A point on a graph that is higher than any other point in its vicinity is called a ***relative maximum***. In other words, the function $y = f(x)$ is said to have a relative maximum at $x=x_0$ if $f(x_0)$ is greater than immediately preceding or succeeding points of the function.

- Similarly, a point on a graph that is lower than any other point in its vicinity is called a ***relative minimum***. In other words, function $y=f(x)$ is said to have a relative minimum at $x = x_0$ if $f(x_0)$ is smaller than immediately preceding or succeeding points of the function.

❖ An extreme point (relative or local extremum) of a function is a point where the function is at a relative maximum or minimum. If the derivative of the function exists at an extreme point, the value of the derivative must be zero, and the tangent line is horizontal.

❖ A point where the derivative equals to zero or undefined is called a ***critical point***. Since the derivative of a function at a relative extreme must be zero, a relative extreme can occur only at a critical point.

❖ A point at which a curve crosses its tangent is called an ***inflection point***. The sign of the first derivative can be positive or negative or can be zero for an inflection point.



P_1, P_5 relative minimum

P_3 relative maximum

P_2, P_4, P_6 inflection points

Determination of maximum and minimum

1. First derivative test
2. Second derivative test

First derivative test – first derivative of the function is used to determine the extreme points.

Necessary condition or first order condition

If the first derivative of a function $y = f(x)$ at point $x = x_0$ is zero i.e. $f'(x) = 0$ then it is an extreme point. It may be a relative maximum or minimum or inflection point.

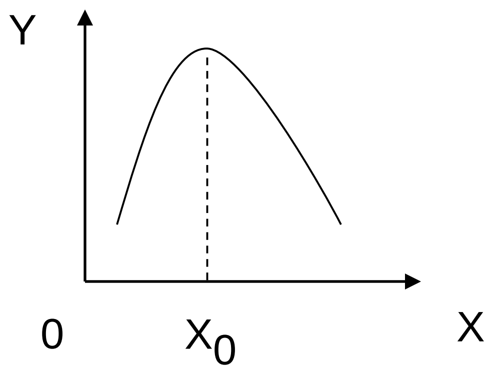
Ex. $y = 2x^2 - 8x + 2$

$$\frac{dy}{dx} = 4x - 8 = 0$$

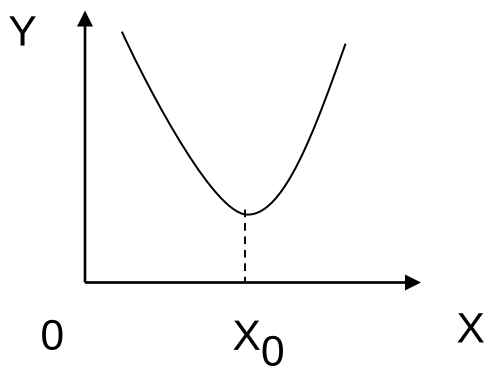
$\therefore$ at $x = 2$ function has either maxima or minima

Sufficient condition or second order condition

- If the sign of the derivative $f'(x)$ changes from positive to negative from the left of the point x_0 to its right, x_0 is a relative maximum.



- If the sign of the derivative $f'(x)$ changes from negative to positive from the left of the point x_0 to its right, x_0 is a relative minimum .



- If $f'(x)$ has the same sign on both sides of point x_0, *f(x)* has neither maximum nor minimum value at $x = x_0$. It is an inflection point.

Find maxima/minima of the following function

$$y = f(x) = x^3 - 12x^2 + 36x + 8$$

$$f'(x) = 3x^2 - 24x + 36 = 0$$

$$x = 2 \quad \text{and} \quad x = 6$$

$$f'(2) = 0 \qquad\qquad f'(6) = 0$$

$\therefore x_1 = 2$ and $x_2 = 6$ are critical points.

$f(2) = 40$ $\quad f(6) = 8$ are stationary values.

Critical points are (2, 40) and (6, 8)

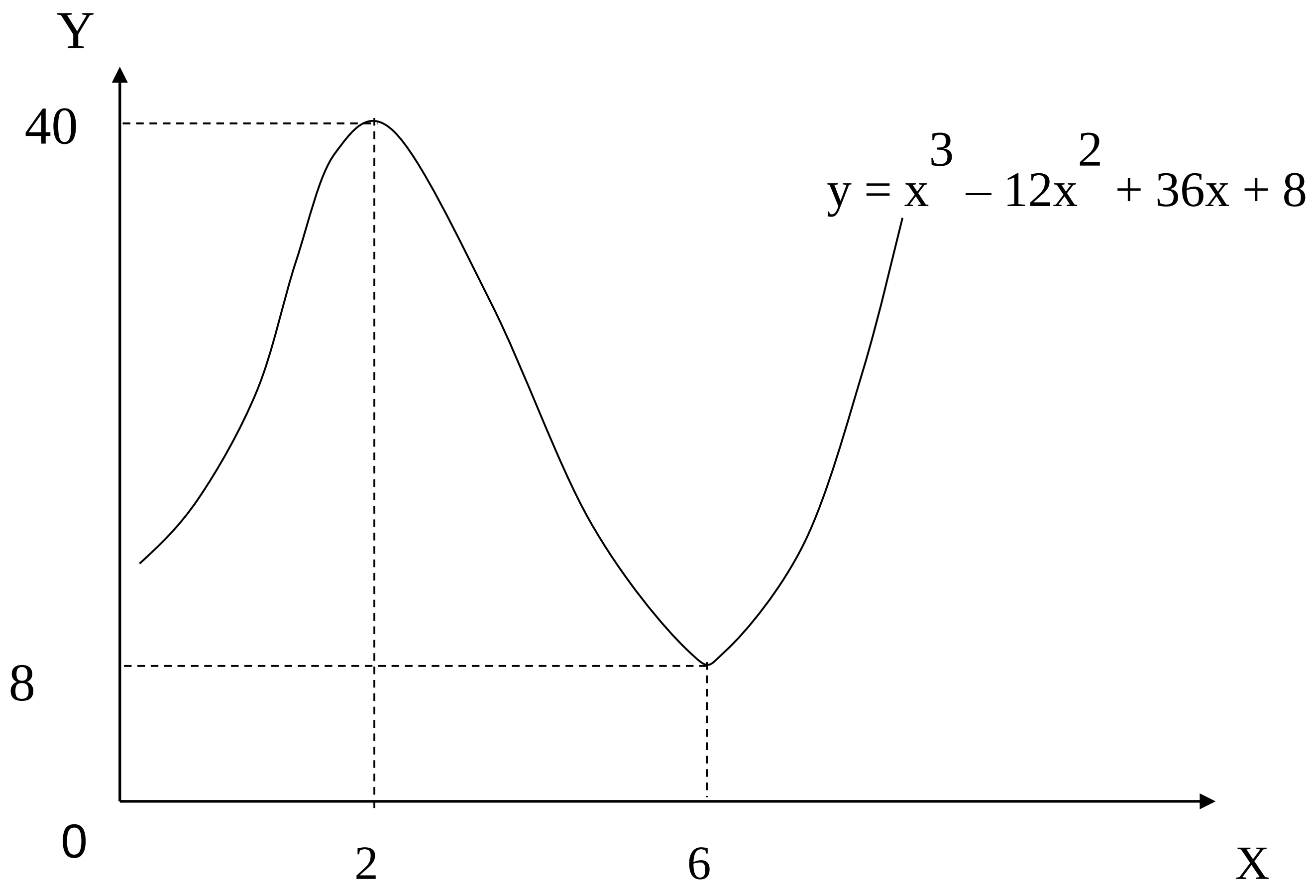
To determine whether $x = 2$ is a maximum or minimum point, substitute two values, one lower than 2 for x, i.e. $x < 2$ and other greater than 2 for x, i.e. $x > 2$ to the $f'(x)$.

At $x = 1$, $f'(x) = f'(1) = 15 > 0$

At $x = 3$, $f'(x) = f'(3) = -9 < 0$

Since the sign has changed from + to -, at $x = 2$ $f(x)$ has a maximum.

What about X = 6?



Y
40
8
0
2
6
X
y = x3 – 12x2 + 36x + 8

Second Derivative Test

Second derivative of the function is used to determine the extreme points.

i Solve $f'(x) = 0$ for the critical values.

ii. For a critical value $x = x_0$

$f(x)$ has a maximum value if $f''(x_0) < 0$

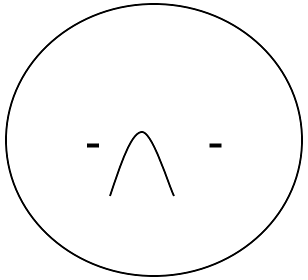
$f(x)$ has a minimum value if $f''(x_0) > 0$

However, the test fails if $f''(x_0) = 0$ or becomes infinite?

i. If at $x = x_0$ $f'(x_0) = 0$, x_0 is a critical point.

ii. If $f''(x_0) < 0$, x_0 is a maximum point.

Maximum

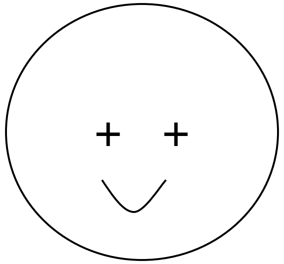


$dy/dx = f'(x) = 0$

$d^2y/dx^2 = f''(x) < 0$

iii. If $f''(x_0) > 0$, x_0 is a minimum point.

Minimum



$dy/dx = f'(x) = 0$

$d^2y/dx^2 = f''(x) > 0$

Ex. $y = f(x) = x^3 - 3x2 + 2$

First order condition $dy/dx = f'(x) = 0$

$dy/dx = f'(x) = 3x^2 - 6x = 0$

Critical values $x_1 = 2, \quad x_2 = 0$

Second order conditions

$d^2y/dx^2 = 6x - 6$

$\left. d^2y/dx^2 \right|_{x = 2} = 6 > 0$

$\therefore$ function has a minimum at $x = 2$

$\left. d^2y/dx^2 \right|_{x = 0} = -6 < 0$

$\therefore$ function has a maximum at $x = 0$

Find relative extrema of the following functions

$y = -x^2 + 4x + 91$

$y = \frac{1}{3}x^3 - 3x^2 + 5x + 3$

If $d^2y/dx^2 = f''(x) = 0$?

In a situation like this,

(1) Examine how the sign of $f'(x)$ change when x substitutes values less than to higher than of x_0

i. If the sign changes + to – at $x = 0$, the function has a maximum.

ii. If the sign changes - to + at $x = 0$, the function has a minimum.

iii. If the sign does not change the function has a inflection at $x = 0$

(2). Take successive higher order derivatives.

Substitute $x = x_0$ to the first non-zero higher order derivative.

- If the value is odd, $x = x_0$ is an inflection point.
- If the value is even, $x = x_0$ is an extreme point.

- If the even number is positive, the point is minimum

- If the even number is negative, the point is maximum.

Example $y = x^5 - \frac{5}{2}x^4 + 17$

1st order condition

$dy/dx = 5x^4 – 10x^3 = 0$

Critical values are $x = 0$, $x = 2$

2nd order condition

Second derivative of y is, $d^2y/dx^2 = 20x^3 – 30x^2$

Now this should be evaluated at $x = 2$ and $x = 0$

$$\left. d^2y/dx^2 \right|_{x=2} = 160 – 120 = 40 > 0$$

Since second derivative of y is greater than zero, at $x = 2$ function has a minimum

Minimum value is $f(2) = 32 – 40 + 17 = 9$

What happened at $x = 0$?

$$\left.\frac{d^2y}{dx^2}\right/_{x=0} = 0$$

To determined exactly the point at $x = 0$,

(1). We can go to first derivative test.

We will substitute -1 and +1 to $f\,'(x)$,

At $x = -1$, At $x = 1$,

$$\frac{dy}{dx} = 15 \qquad \frac{dy}{dx} = -15$$

Since the sign has changed from .+ to - , at $x = 0$ the function has a maximum

(2). Evaluate successive higher order derivatives.

Evaluate 3rd derivative at $x = 0$,

$$d^2y/dx^2 = 20x^3 - 30x^2$$

at $x = 0$ $\quad d^2y/dx^2 = 0$ *cannot decide!*

Evaluate 3rd derivative at x = 0,

$$d^3y/dx^3 = 60x^2 - 60x$$

at $x = 0$ $\quad d^3y/dx^3 = 0$ *cannot decide!*

Evaluate 4th derivative at x = 0

$$d^4x/dx^4 = 120x - 60$$

at $x = 0$ $\quad d^4x/dx^4 = -60 < 0$

Since the value is even and is negative, at $x=0$ the function has a maximum.

Extreme Values of a Function with Two Choice Variables

$z = f(x, y)$

Condition	Maximum	Minimum
First order or Sufficient	$f_x = 0,\ f_y = 0$	$f_x = 0,\ f_y = 0$
Second order or Necessary	$f_{xx}, f_{yy} < 0$ and $f_{xx} f_{yy} > f_{xy}^2$	$f_{xx}, f_{yy} > 0$ and $f_{xx} f_{yy} > f_{xy}^2$
	If f_{xx} and f_{yy} have opposite signs, a saddle point If $f_{xx} f_{yy} < f_{xy}^2$ *a saddle point*	

(1). The total cost of producing a given commodity is

$TC = \frac{1}{4} x^2 + 30x + 25$ and the price of the commodity is

$P = 60 - \frac{1}{2} x$.

Find the level of output which yield the maximum profit

(2). Cost function of a perfectly competitive firm is

$$TC = 1/3\ q^3 – 5q^2 + 30q + 10$$

If price p = 6, find the profit maximizing output level.

ex. Find the extreme value(s) of $z = 8x^3 + 2xy - 3x^2 + y^2 + 1$

First order condition is $f_x = 0$ and $f_y = 0$

$$f_x = \partial z/\partial x = 24x^2 + 2y - 6x = 0 \qquad (1)$$

$$f_y = \partial z/\partial x = 2x + 2y = 0 \qquad (2)$$

From (2) $\quad y = - x$

Substituting into (1) $\quad 24x^2 - 8x = 0$

$$3x^2 - x = 0$$

$$x(3x - 1) = 0$$

$x_1 = ⅓, \qquad y_1 = - ⅓$

$x_2 = 0 \qquad y_2 = 0$

For second order condition

$f_{xx} = 48x - 6$ $\quad f_{yy} = 2$ $\quad f_{xy} = 2$ $\quad f_{yx} = 2$

When $x = 0$

$f_{xx} = -6$ $\quad f_{yy} = 2$

Since the signs of f_{xx} and f_{yy} are opposite, the product of them yield a negative value. Obviously it is less than f_{xy}^2, hence failed the second order condition for maxima and minima.

When $x = 1/3$

$f_{xx} = 10$ $f_{yy} = 2$ $f_{xy} = 2$

The product $f_{xx}f_{yy} > f_{xy}^2$. Thus, second order condition satisfies for a minimum. Point $x = 1/3$ and $y = -1/3$, i.e. $(1/3, -1/3)$ is a minimum and minimum value of the function is 23/27.

Economic applications

Profit maximization

Profit (Π) = Total Revenue (R) – Total Cost (C)

$$\Pi = R - C$$

$$R = f(q) \quad \text{and} \quad C = f(q)$$

$$\Pi(q) = R(q) - C(q)$$

For profit maximization, the difference between R(q) and C(q) should be maximized.

First order condition

$d\Pi/dq = \Pi' = 0$

$\Pi'(q) = R'(q) – C'(q)$

$= 0$ *iff* $R'(q) = C'(q)$

Thus, at the output level which yield maximum profit,

$R'(q) = C'(q)$

$MR = MC$

This is the first order condition for profit maximization.

Second order condition

$$d^2\Pi/dq^2 = \Pi''(q) < 0$$

$$\Pi''(q) = R''(q) - C''(q)$$

$$< 0 \quad \textit{iff}\ R''(q) < C''(q)$$

Thus, to satisfy the second order condition R''(q) < C''(q).

R''(q) = the rate of change of MR

C''(q) – the rate of change of MC

At the output level which MC = MR, and R''(q) < C''(q) profit is maximized.

The demand and cost functions of a firm is given respectively as P = 32 – Q and C = 21Q + 24.

i. Find the output level that maximize the total revenue (TR).

ii. Find the output level that maximize the profit (Π).

i. TR = 32Q – Q2

$$\frac{dTR}{dQ} = 32 - 2Q = 0$$

$$\therefore \qquad Q = 16$$

For the second order condition,

$$\frac{d^2TR}{dQ^2} = -2 < 0$$

$\therefore$ Q =16 gives a relative maximum

TR = 32Q - Q^2 = 256

ii. $\Pi = 32Q - Q^2 - 21Q - 24$

$= -Q^2 + 11Q - 24$

$$\frac{d\Pi}{dQ} = -2Q + 11 = 0$$

$$Q = 5.5$$

For the second order condition

$$\frac{d^2\Pi}{dQ^2} = -2 < 0$$

$\therefore$ $\text{Q} = 5.5$ provides a relative maximum

$$\Pi = 6.25$$

The profit function of a firm is given as

$$\pi = -\frac{1}{3}q^3 + 8q^2 - 39q - 50$$

Find the output level that maximize the profit.

For the 1st order condition

$$\frac{d\pi}{dq} = -q^2 + 16q - 39 = 0$$

q = 13 and q = 3

For the 2nd order condition $$\frac{d^2\pi}{dq^2} = -2q + 16$$

At q = 13,

$$\frac{d^2\pi}{dq^2} = -10 < 0$$

At q = 3, $$\frac{d^2\pi}{dq^2} = 10 > 0$$

Thus q = 13 fulfills the 2nd order condition for profit maximization.

1. Determine maxima and/or minima of the following function:

$$y = -\frac{1}{3}q^3 + 8q^2 - 39q - 50$$

2. The total cost of producing a given commodity is

   $TC = \frac{1}{4}x^2 + 35x + 25$ and the price of the commodity is $P = 50 - \frac{1}{2}x$.

   i. Find the level of output which yield the maximum profit
   ii. Show that average cost is minimum at this output level.

3. Cost function of a perfectly competitive firm is $TC = 1/3\ q^3 - 5q^2 + 30q + 10$

   If price $p = 6$, find the profit maximizing output level.

4. For a new product, a manufacturer spends Rs. 100,000 on the infrastructure and the variable cost is estimated as Rs.150 per unit of the product. The sale price per unit was fixed at Rs.200. Find the break-even point.

5. A manufacturing company finds that the daily cost of producing x items of a product is given by $C(x) = 210x+7000$.

(i) If each item is sold for Rs. 350, find the minimum number that must be produced and sold daily to ensure no loss.

(ii) If the selling price is increased by Rs. 35 per piece, what would be the break-even point?

6. The cost function for x units of a product produced and sold by a company is $C(x)=2500+0.005x^2$ and the total revenue is given as $R = 4x$. Find how many items should be produced to maximize the profit. What is the maximum profit?

7. If the total cost function C of a product is given by $C = 2x\left[\frac{x+7}{x+5}\right]+7$

Prove that the marginal cost falls continuously as the output increases.

8. Total cost function of a firm is given as $TC = \frac{1}{3}q^3 - 4.5q^2 + 14q + 22$ Find the output level that minimize total cost (TC).

9. Total cost function and demand functions of a firm respectively are

$TC = \frac{1}{3}q^3 - 8.5q^2 + 50q + 90$ and 22-0.5q-P = 0.

Find the output level that maximize the total revenue (TR) and the profit (Π).

10. Short production function of a firm is $Q = 6L^2 - 0.4L^3$

Q is the output and L is labor input.

i. Find average product (AP) and marginal product (MP) of labor input.

ii. Find the quantity of labor which maximize output'

iii. Find the level of output which maximize average product of labor.

11. A Company produced a product with Rs 18000 as fixed costs. The variable cost is estimated to be 30% of the total revenue when it is sold at a rate of Rs.20 per unit. Find the total revenue, total cost and profit functions.

12. The total revenue received from the sale of x units of a product is given by

$R(x) = 12x + 2x2 + 6$.

Find (i) the average revenue

(ii) the marginal revenue

(iii) marginal revenue at $x = 50$

(iv) the actual revenue from selling 51st item