Integration

Integration is the opposite process of differentiation. Given the derivative of a function we can find the primitive function by the process of integration.

Suppose F(x) is a function of x.

The derivative of F(x) = $\dfrac{d[F(x)]}{dx} = f(x)$

Where, F(x) is the integral of $f(x)$ and

$f(x)$ is the integrand

Integration of $f(x)$ can be written as

$$\int f(x)dx = F(x) + c$$

dx shows that integration is being done by x

- The symbol $\int$ is the integral sign, *f(x)* is integrand (අනුකලයය) and *c* is a constant of integration.

- Constant *c* may have different values and accordingly we can have different members of *f(x)dx* family.

For example, if $y = x^4$, the derivative of *y* is $\frac{dy}{dx} = 4x^3$

- When $\frac{dy}{dx} = 4x^3$ is given, to find *y* we have to follow the opposite process of differentiation. Thus, $\int 4x^3 dx = x^4$

However, $\frac{dy}{dx} = 4x^3$ may be the derivative of different differentiable functions of x.

For example, it is equivalent to the derivatives of

$y = x^4+1$, $y = x^4 - 5$, $y = x^4 + a$ etc.

- Thus, if we add a constant (c) to the integral it will match with the differential function.

$$\therefore \quad \int 4x^3 dx = x^4 + c$$

Integration is two types:

i. Definite integration: integral is a definite numerical value.

ii. Indefinite integration: integral is not a definite numerical value but a function of *x*.

Rules of integration

Rule 1. Constant function rule

i. $\int k dx = kx + c$ $\qquad \because dkx/dy = k$

ii. $\int 8 dx = 8x + c$

iii. $\int dx = x + c$

iv. $\int - dx = - x + c$

Rule 2: the Power Rule

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \qquad ; n \neq -1$$

e.g. (i) $\int x^5 dx = \frac{1}{6} x^6 + c$

e.g. (ii) $\int \frac{1}{x^4}\, dx =$

e.g.(iii) $\int \sqrt{x^3}\, dx = \int x^{\frac{3}{2}} dx = ?$

Rule 3. the Integral of a Multiple

$$\int kf(x)dx = k\int f(x)dx$$; k is a constant

e.g. (1) $\int 2x^3 dx = 2\int x^3 dx = \frac{1}{2}x^4 + c$

e.g. (2) $\int 6y^4 dy = ?$

e.g. (3) $\int 16x^{-3} dx =$

Rule 4.the Integral of a Sum or Difference

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

e.g. (1) $\int (2x^2 - 5x + 3)dx = 2\int x^2 dx - 5\int x dx + 3\int dx$

$Ex.(1)\ \int \frac{3x^2 - 4x + 1}{x} dx$

$Ex\ (3)\ \int \sqrt{x}(3 - 4x)dx$

$Ex.\ (2)\ \int \frac{x^2 + 1}{\sqrt{x}} dx$

Rule 5. Generalized Power Function Rule

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a} + c$$

Ex. i. $\int (4x+2)^{1/2} dx = ?$

Ex. ii. $\int (x^3+1)^2 dx = ?$

Ex. iii. $\int \frac{3}{\sqrt{4-5x}} dx$

Ex. iv. $\int (2x+1)^{1/3} dx$

Rule 6. the Logarithmic Rule

Rule 6.1 $$\int \frac{1}{x}\,dx = ln|x| \qquad x \neq 0$$

i. $$\int \frac{3}{x}\,dx = 3\int \frac{1}{x}\,dx = 3\ln|x| + c$$

ii. $\int \frac{3}{2x-1}\,dx = ?$

iii. $\int \frac{5}{(2-7x)}\,dx = ?$

Rule 6.2

$$\int \frac{1}{f(x)}\, dx = \frac{\ln[f(x)]}{f'(x)} + c$$

e.g. (i) $$\int \frac{1}{ax + b}\, dx = \frac{\ln(ax+b))}{a} + c$$

Ex. (1) $$\int \frac{1}{(5x+3)}\, dx = ?$$

Ex. (2) $$\int \frac{1}{(2x^2+5x+2)}\, dx = ?$$

Rule 6.3 If $f(x)$ is a function of x and its differential coefficient with respect to x is $f'(x)$. The derivative of $\ln[f(x)]$ is

$$\frac{d\ln[f(x)]}{dx} = \frac{f'(x)}{f(x)}$$

$$\therefore \quad \int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right|$$

e.g. i. $\int \frac{2x}{x^2+1}dx = \ln\left|x^2+1\right|$

Ex (1) $3\int \frac{x}{x^2+2}dx =$

Ex (3) $2\int \frac{x^2+2x}{(2x^3+x^2+1)}dx = ?$

Ex. (2). $\int \frac{x^2}{x^3+1}dx$

Rule 7: the Exponential Rule

Rule 7.1. $$\int e^x dx = e^x + c$$

e.g. $\int(e^x+2x)dx = e^x + x^2+c$

Rule 7.2 $$\int e^{f(x)} dx = \frac{e^{f(x)}}{f'(x)} + c$$

e.g. $$\int (e^{4x} - 3x^2 + 2e)dx = \int e^{4x}dx - 3\int x^2 dx + 2e\int dx$$
$$= \frac{1}{4}e^{4x} - x^3 + 2ex + c$$

Ex. (1) $\int e^{4x+3} dx = ?$

Ex. (2) $$\int \frac{e^{3x} + e^{\frac{1}{2}x} - e}{e^x} dx = ?$$

Rule 7.3 $$When \;\; y = e^{f(x)}, \quad \frac{d[e^{f(x)}]}{dx} = e^{f(x)} f'(x)$$

$$\therefore \qquad \int e^{f(x)} f'(x) dx = e^{f(x)} + c$$

Ex. (i) $$\int e^{2x^2} 4x \, dx = e^{2x^2} + c$$

e.g. (ii). Find indefinite integral of $\int xe^{x^2}dx$

$f(x) = x^2$ and $f'(x) = 2x$

To apply the above rule we have to adjust the original function as,

$$\frac{1}{2}\int 2xe^{x^2}dx$$

There is not a fundamental difference between the original function and this function.

We can apply the above rule for this function

$$\int xe^{x^2}dx = \frac{1}{2}\int 2xe^{x^2}dx = e^{x^2} + c$$

e.g. (iii). $\int xe^{x^2+3}dx$

$$f(x) = x^2 + 3 \quad \text{and} \quad f'(x) = 2x$$

Rule 8: the Substitution rule

e.g. (i) Evaluate $\int 2x(x^2+1)dx$

Let $u = x^2+1$; then $du/dx = 2x$ or $dx = du/2x$.

Now $du/2x$ can be substituted for dx of the above function.

$$\therefore \quad \int 2x(x^2+1)dx = \int 2xu\frac{du}{2x} = \int u\,du$$

$$= \frac{1}{2}u^2 + c$$

$$= \frac{1}{2}(x^2+1)^2 + c$$

$$= \frac{1}{2}(x^4+2x^2+1) + c$$

e.g. ii $\int 2x(x^2+8)^3\,dx$

Ex. (1) $\int (x^3+2)^2 3x^2 dx = ?$

Ex. (2) $\int (x^3+2)^{\frac{1}{2}} x^2 dx = \frac{1}{3}\int (x^3+2)^{\frac{1}{2}} 3x^2 dx = ?$

Integration by Parts

When $u=f(x)$ and $v=g(x)$

$$y = uv$$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

By integrating both sides with respect to x

$$\int \frac{d(uv)}{dx}dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$uv = \int u\,dv + \int v\,du$$

$$\int u\,dv = uv - \int v\,du$$

The integral of u with respect to v is equal to uv less the integral of v with respect to u.

e.g. $\int xe^{x}dx$

We will defined $x = u$ and $dv = e^{x}dx$

$$\frac{du}{dx} = 1 \qquad\qquad Since\;\; dv = e^{x}dx, \qquad v = e^{x}$$

$$\therefore \quad du = dx$$

$$\int xe^{x}dx = xe^{x} - \int e^{x}du$$

$$= xe^{x} - \int e^{x}dx \qquad \Theta \quad du = dx$$

$$= xe^{x} - e^{x} = e^{x}(x-1) + c$$

The Definite Integral

The indefinite integral of a continues function *f(x)* is:

$$\int f(x)dx = F(x) + c$$

If we choose two values of x in the domain, say a and b (b > a), substitute them successively into the right side of the above equation and form the difference we get a numerical value that is independent of the constant c.

$$[F(b)+c]-[F(a)+c] = F(b)-F(a)$$

This value is called the definite integral of $f(x)$ from a to b. a and b are lower and upper limits of integration, respectively.

Now, we will modify the integration sign to indicate the definite integral of $f(x)$ from a to b as:

$$\int_a^b f(x)dx = [F(x)+c]_a^b = [F(b)+c] - [F(a)+c]$$

$$= F(b) - F(a)$$

Evaluate the following definite integrals

1. $\int_2^4 3x^2 dx$=?

2. $\int_1^2 (6x^2 + 8x + 1)dx = ?$

3. $\int_0^2 (x+7)^3 dx = ?$

4. $\int_a^b ke^x dx = ?$

5 $$\int_0^4 \left(\frac{1}{1+x} + 2x \right) dx = \left[\ln|1+x| + x^2 \right]_0^4$$

$$= [\ln 5 + 16] - [\ln 1 + 0]$$

$$= \ln 5 + 16 \qquad \Theta \ \ln 1 = 0$$

Basic Properties of Definite Integral

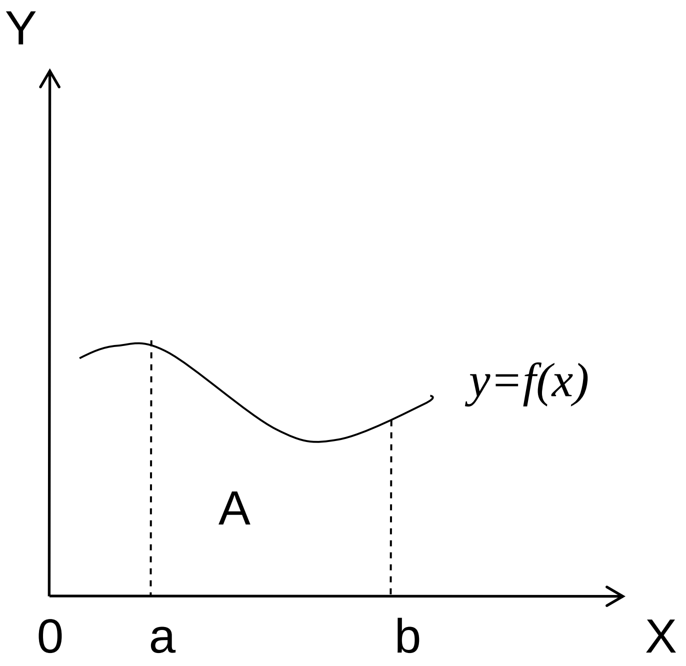
$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx \qquad \text{where } a < b < c$$

The Definite Integral as an Area

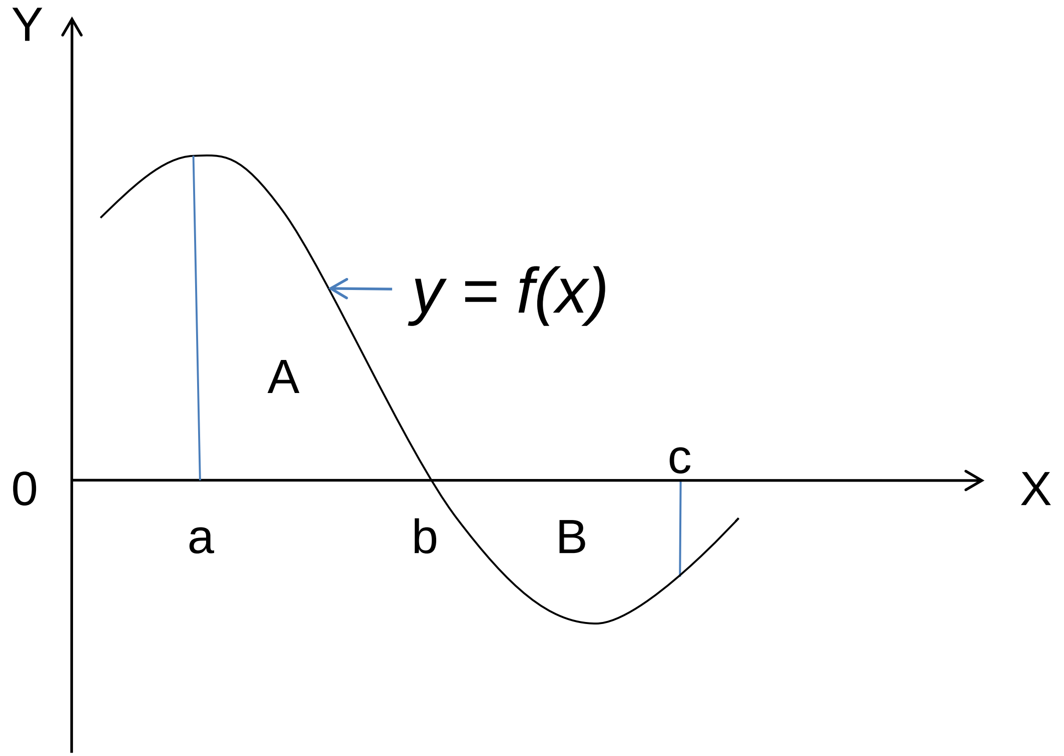
The area of a region bounded by the curve $y = f(x)$, and by the x axis, on the left by $x = a$, and on the right by $x = b$ is given by,



$$\text{Area } (A) = \int_a^b f(x)dx$$

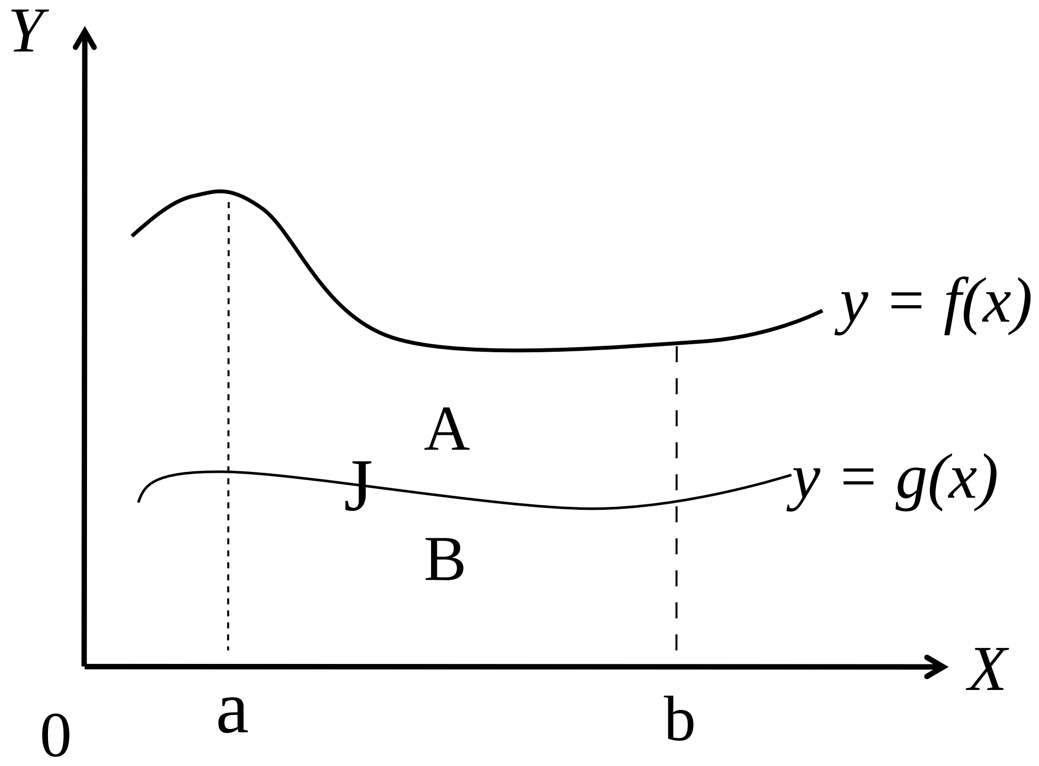
If the curve $y = f(x)$ lies below the x axis, then

$$\text{Area } (A) = -\int_a^b f(x)dx$$



The area (A) + B) = $\int_{a}^{b} f(x)dx + \int_{b}^{c} f(x)dx$

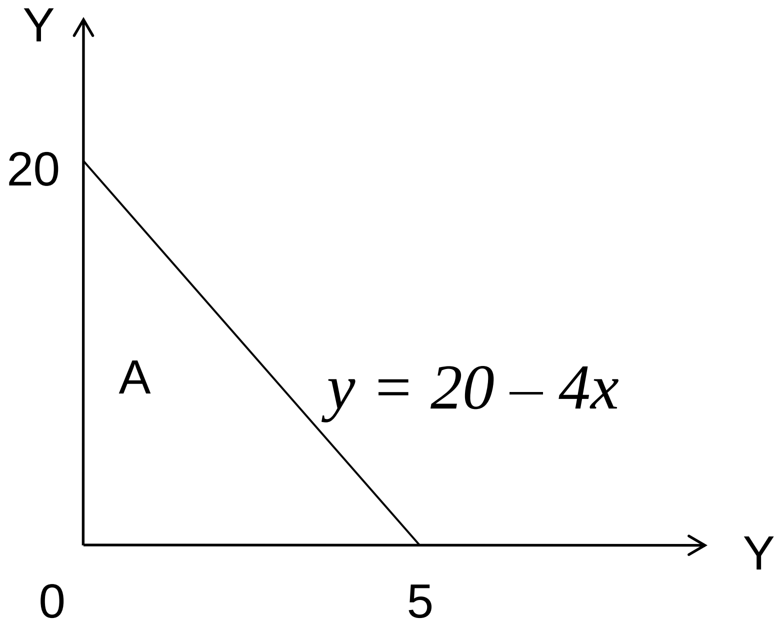
IF *f(x)* and *g(x)* are the two functions of *x* and *f(x)* > *g(x)*



Area (A) = Area (J) – Area (B)

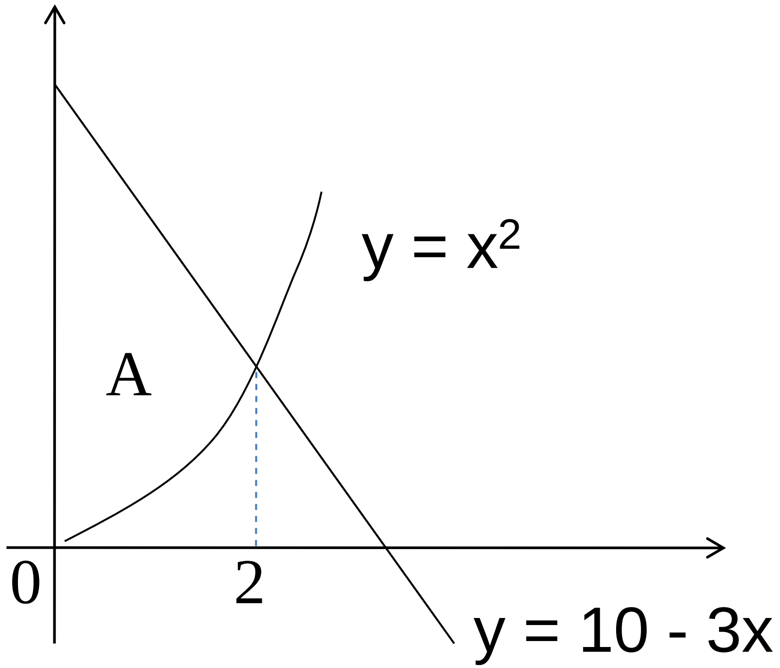
$$= \int_a^b f(x)dx - \int_a^b g(x)dx$$

e.g.1. Determine the area under the curve given by the function $y = 20 - 4x$ over the interval 0 to 5.



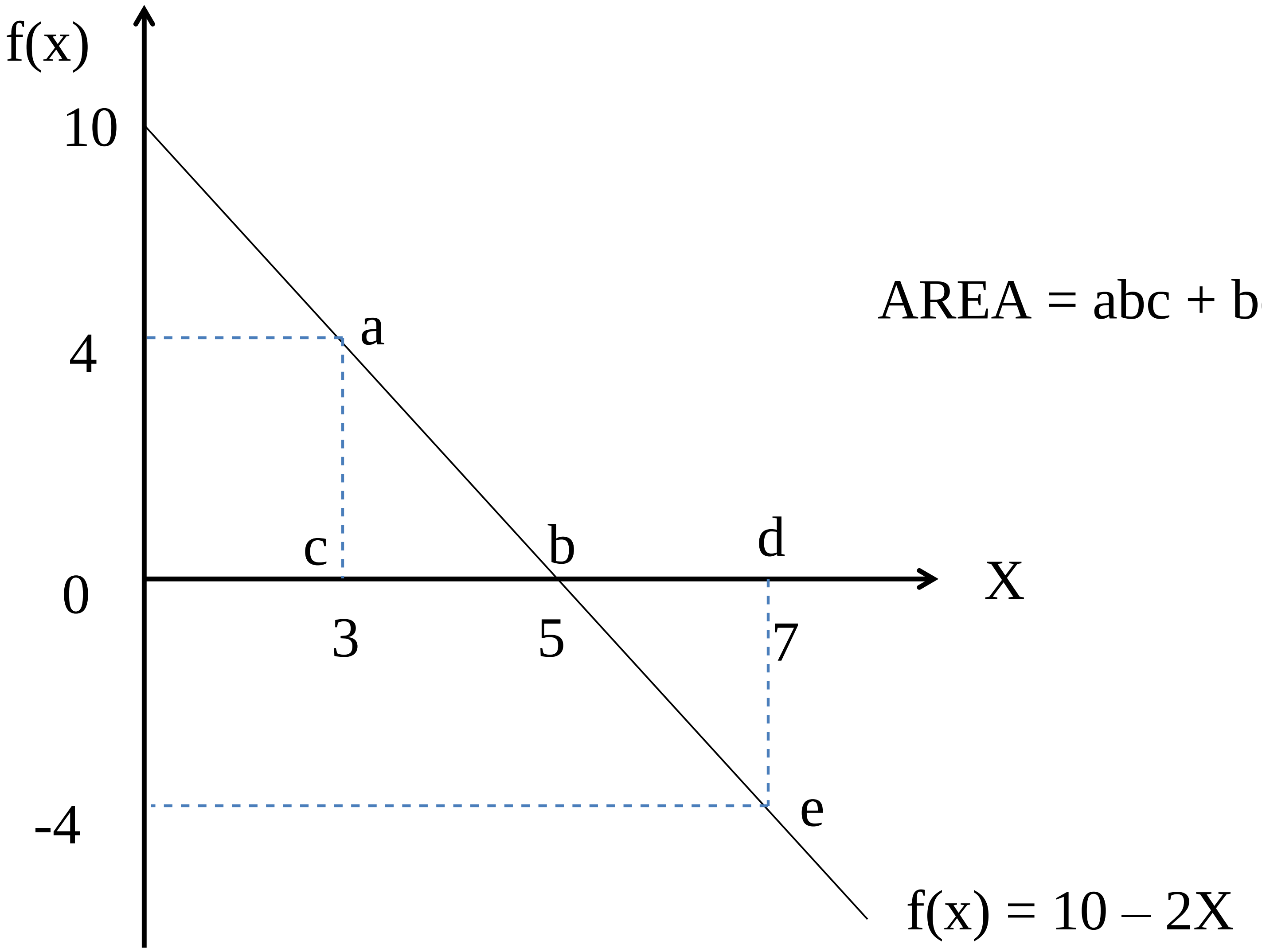
Area (A) =

e.g. 2 Find the area bounded by the functions $y = x^2$, and $y = 10 - 3x$ and Y axis.



$$A = \int_0^2 (10 - 3x)\,dx - \int_0^2 x^2\,dx$$

3. Determine the area between the curve of $f(x) = 10 - 2X$ and the X axis for values of X = 3 to X = 7.



AREA = abc + bde =?

Multiple Integral

The integral of a function that include more than one variable is defined as *Multiple integral*.
Indefinite integral of a two variable function is given by,

$$\iint f(x,y)\,dy\,dx$$

Definite integral of a two variables (x and y) function is given by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx$$

e.g. If the joint probability density function of variables x and y is $f(x,y) = (x+y)$. Find the probability p = $(0<x<½ , 0<y<¼)$

$$\int_0^{1/4}\int_0^{1/2}(x+y)\,dx\,dy = \int_0^{1/2}[(xy+\frac{y^2}{2})dx]_0^{1/4}$$

$$= \int_0^{1/2}(\frac{x}{4}+\frac{1}{32})dx$$

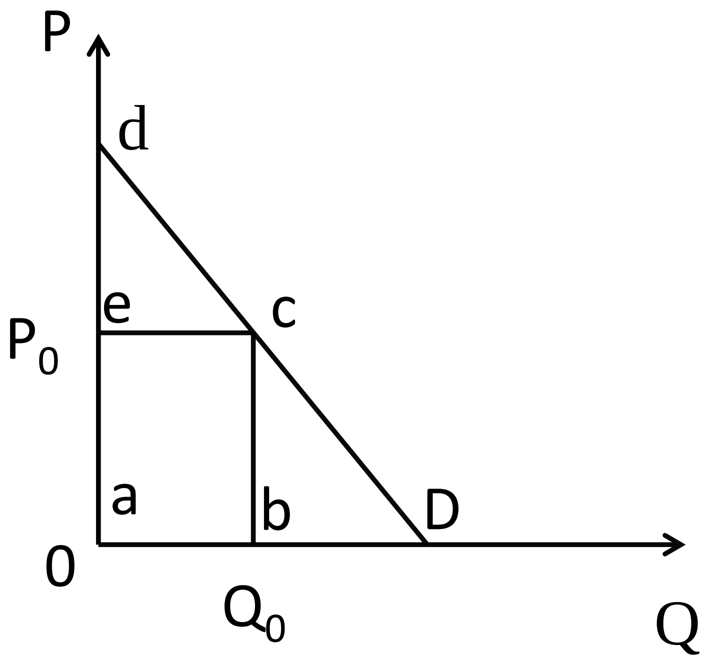
$$\int_0^{1/2}\left(\frac{x}{4}+\frac{1}{32}\right)dx=\left[\frac{x^2}{8}+\frac{x}{32}\right]_0^{1/2}$$

$$=\left[\frac{1}{32}+\frac{1}{64}\right]-[0]$$

$$\int_0^{1/4}\int_0^{1/2}(x+y)\,dx\,dy=\frac{3}{64}$$

Consumer surplus

Consumer surplus is the difference between what a consumer pays for an item and what he is willing to pay. It (CS) measures the net benefit that a consumer enjoys from purchasing a particular commodity in the market.



Demand function describes willingness to pay for an item. Equilibrium price and quantity are P_0 and Q_0 respectively. Consumer is willing to pay a higher price than P_0, for all quantities less than Q_0. However, he actually pays only P_0 price for all units.

The ‘total value’ of their purchases is measured by the area *abcd*. The ‘total expenditure’ is given by area *abce*. The difference between total value and the total expenditure is the surplus value that they are not paying.

Integral calculus can be used to estimate consumer surplus.

e.g. (1) Consumer's demand function for an item has been estimated to be

$$P = 30 - 2Q$$

Where, P is the unit price (Rs.)

Q is the monthly per capita consumption of the item (kg per month).

Determine the total expenditure and consumer surplus when the price per unit is Rs 5.

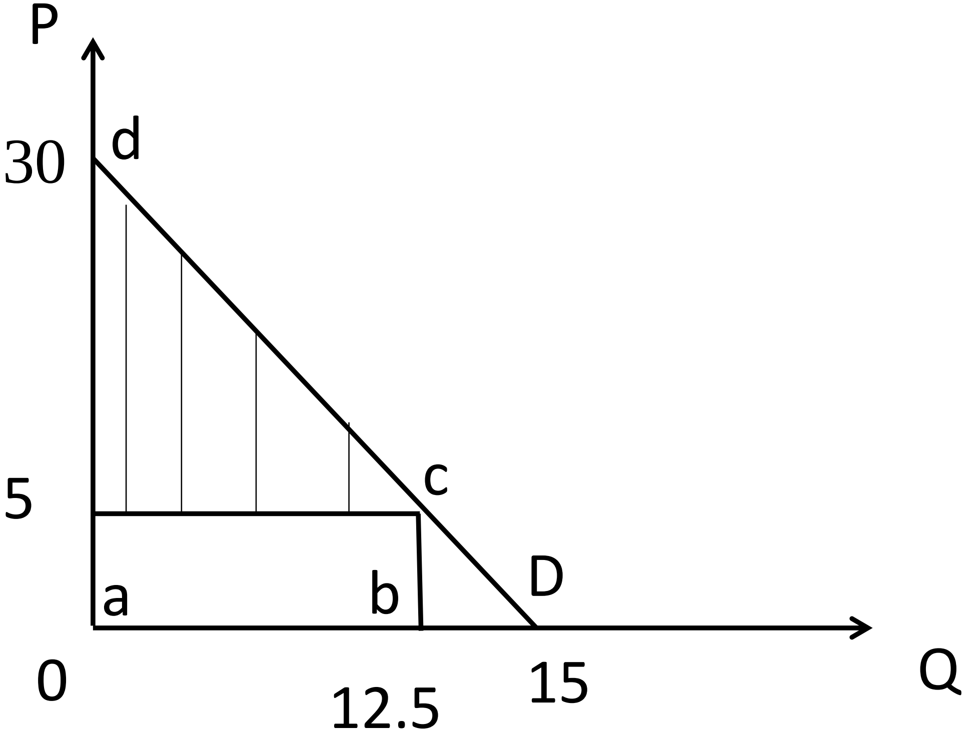
Quantity demanded,

$P = 30 - 2Q$

$$Q = \frac{30 - p}{2}$$

When P = 5

Q = 12.5 units



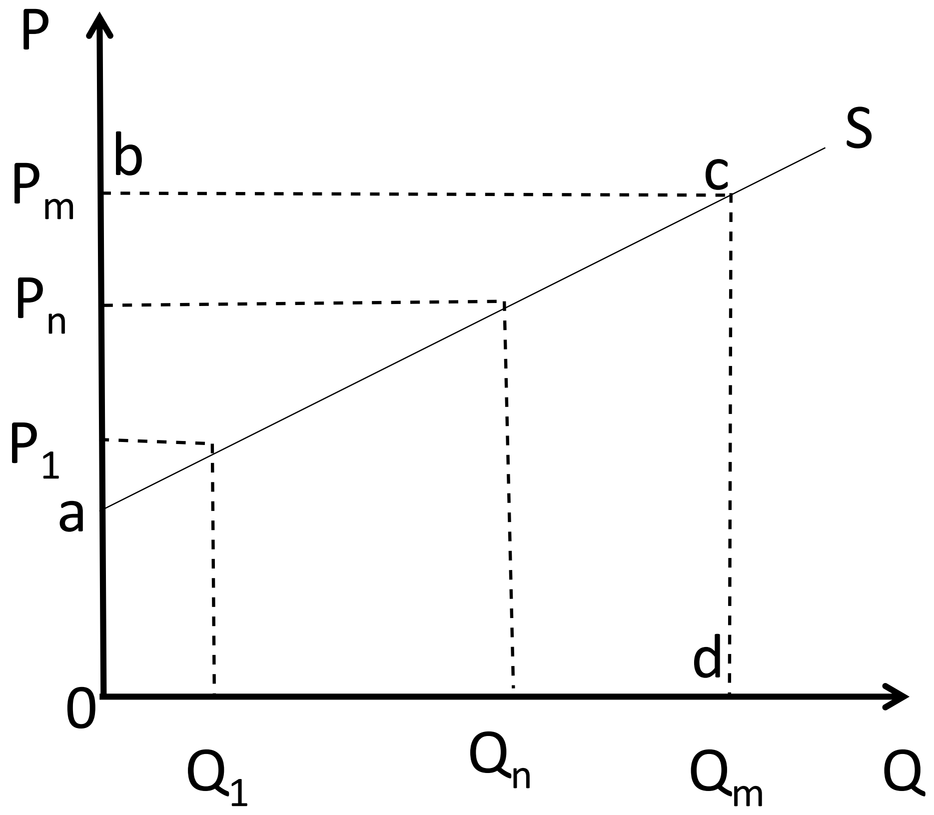
Total expenditure TE = ?

Total value of purchases TV=?

Consumer surplus CS = TV-TE

Producer surplus

A similar concept to consumer surplus applies to producer. It is called ‘producer surplus’. It measures the payments that producers receive for a quantity of the product than what they expect.



Supply curve of the figure shows the locus of minimum prices that must be paid to induce producers to supply different quantities of their products.

For example, to provide Q_1 they expect P_1 price and to provide Q_n they expect P_n price. However, they receive market price P_m for all units (Q_m) placed on the market.

Producers’ total revenue is the product of P_mQ_m or the area 0bcd. The expected revenue to provide Qm equal to the area 0acd. Producer surplus is the difference between are 0bcd and 0acd. i.e. area cbc.

e.g. The supply function of a producer which supply a certain product is given by $P = 1000 + 50Q$. Where P is the unit price and Q is the quantity supplies each day.
Determine producer surplus if he sell the product for Rs 2000 each.

Economic Applications

1. If the marginal cost (MC) function of a firm is $C' = 2e^{0.2Q}$ and the fixed cost $C_F = 90$. Find the total cost function (TC).

2. If the marginal cost (MC) function of a firm is $MC(q) = (6q^2 + 4)\sqrt{2q^3 + 4q + 36}$ and fixed cost of the function is 1088. Find the total cost function (TC).

3. If the marginal saving function of a country is $S'(Y) = 0.3 - 0.1Y^{-1/2}$. If the aggregate saving S is zero when income (Y) is 81. Find the saving function S(Y).

4. Consumer's demand function for a given commodity has been estimated to be $P = 30 – 2Q$

where, P is the price of a unit of the commodity and Q is the per capita consumption of the commodity per person per month. Determine (a) the total expenditure and (b) the consumer surplus when the price of a unit is 5.

5. If the supply function of a commodity is $P = 1000 + 50Q$

where, P is the price per unit and Q is the number of units sold each day. Find the producer surplus when the price of a unit of the commodity is 2000.

6. If the willingness of a nurse to provide her service is defined by the supply function $W = 2.5 + 0.5H$

where, W is the wage rate per unit

H is hours of work provided each week.

Determine the producer surplus paid to the nurse if the prevailing wage rate is 9 per hour.

7. At a certain factory, the marginal cost is $3(q - 4)^2$ dollars per unit when the level of production is q units. By how much will the total manufacturing cost increase if the level of production is raised from 6 units to 10 units?

8. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t) = 50 + t^2$ hundred dollars per year, while a second investment will be generating profit at the rate of $P_2'(t) = 200 + 5t$ hundred dollars per year. $P_1(t)$ and $P_2(t)$, satisfy $P_2(t) \geq P_1(t)$ for the first N years ($0 \leq t \leq N$).

(a) For how many years does the rate of profitability of the second investment exceed that of the first?

(b) Compute the net excess profit for the time period determined in part (a). Interpret the net excess profit as an area.

9. Suppose that when it is t years old, a particular industrial machine generates revenue at the rate $R(t) = 5{,}000 - 20t^2$ dollars per year and that operating and servicing costs related to the machine accumulate at the rate $C(t)$ $2{,}000 + 10t^2$ dollars per year

(a) How many years pass before the profitability of the machine begins to decline?

(b) Compute the net earnings generated by the machine over the time period determined in part (a)

10. The demand and supply functions of a competitive market are as follows;

. $D(q) = \dfrac{24}{q+2}$ and $S(q) = -8 + q$. Find consumer and producer surplus.

(11) A tire manufacturer estimates that q (thousand) radial tires will be purchased (demanded) by wholesalers when the price is

$$p = D(q) = -0.1q^2 + 90$$

dollars per tire, and the same number of tires will be supplied when the price is

$$p = S(q) = 0.2q^2 + q + 50 \quad \text{dollars per tire.}$$

(a) Find the equilibrium price (where supply equals demand) and the quantity supplied and demanded at that price.

(b) Determine the consumers' and producers' surplus at the equilibrium price.